Technical Report: NAVTRADEVCEN 1205-2

SIMULATION OF HELICOPTER AND V/STOL AIRCRAFT

VOLUME II

V/STOL ANALYSIS REPORT

Study, Equations of Motion of Vertical/Short Take-Off and Landing Operational Flight/Weapon System Trainers

Walter McIntyre

119 p

$0.75

Melpar, Inc.
Falls Church, Virginia

September 1963

Prepared for

U. S. NAVAL TRAINING DEVICE CENTER

Port Washington, New York

Contract No. N61339-1205

SIMULATION OF HELICOPTER AND V/STOL AIRCRAFT
VOLUME II
V/STOL ANALYSIS REPORT

ABSTRACT

The report promotes an understanding of V/STOL analysis for simulation purposes and develops equations of motion compatible to either analogue or real time digital solution.

A general set of equations of motion are developed in which axis systems and aerodynamic coefficients are minimized. Equations of motion are then developed for five different V/STOL aircraft wherein the need for additional axis systems and aerodynamic coefficients for a particular V/STOL configuration is developed.

NATRADEVCEN 1205-2

SIMULATION OF HELICOPTER AND V/STOL AIRCRAFT
VOLUME II
V/STOL ANALYSIS REPORT

Foreword

With increasing interest in Vertical/Short Take Off and Landing (V/STOL) aircraft by the military services it was determined that it would be to the advantage of the U.S. Naval Training Device Center to gather together in one report a study of axis systems used for representation of such aircraft, and to develop general aerodynamic equations of motion which may be used in simulation of these aircraft. This effort has been accomplished in this report, for five experimental categories of V/STOL aircraft, each of which, as of this date, appear to give promise of becoming operational.

Carmine C. Castellano

Carmine C. Castellano
Aerospace Staff Conultant
U.S. Naval Training Device Center

TABLE OF CONTENTS

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SECTION I

INTRODUCTION

The purpose of this report is to promote an understanding of V/STOL analysis for simulation purposes and to develop equations of motion in a form most compatible to either analog or real-time digital solution. Only V/STOL aircraft configurations are considered in this analysis.

In Section II of the report a method of attack for the general V/STOL problem is developed and various axis systems are selected. The method developed is then applied to a number of different types of V/STOL aircraft. A model and its conceptual representation are developed in Section III. This model may be used as a guideline in developing equations of motion for various types of V/STOL aircraft but cannot be further simplified in general. The various terms in the final equations for a particular aircraft configuration must be analyzed for the magnitude of their contributions before final simplification is possible.

The XC-142A tilt-wing and the VZ-4DA tilt-duct aircraft are used to illustrate equation development in Section IV. Equations are presented though not painstakingly developed for the X-19 tilt-prop, the XV-5A fan-in-wing and the P.1127 rotating thrust. A great deal of the symbology used for these different V/STOL aircraft is used just as presented by the respective manufacturer in order that verifying data might be available in the proper form at the earliest possible time. At present it appears that the equations developed will comprise a minimum requirement. Terms in these equations which prove in flight test to be sufficiently small, will be discarded. Analysis of flight test data may enable simulation with less rigorous equations but no such simplification can now safely be made.

A bibliography and Appendix A dealing with basic concepts of axis systems and equations of motion have been provided for the convenience of the reader.

(1) Refer to NAVTRADEVCEN 1205-1 for an analysis of single and tandem rotor helicopter configurations.

SECTION II

AXIS SYSTEMS

METHOD OF SELECTION

We will initially consider that the aircraft is operating out of the V/STOL region. This aircraft whether it be a transport, observation plane, subsonic interceptor or any similar type of vehicle can be considered to be operating in a conventional flight regime. A description of both the equations of motion and necessary axis systems for this mode of operation has been done, notably by Connelly and others in References 2 and 7. Connelly, in Reference 2, discusses simulation of fixed-wing aircraft and presents optimum representations of the equations of motion which may be used for construction of a computer. Following the recommendations of Reference 2, we will use wind axes for the description of forces and aircraft body axes for the description of moments in the conventional flight regime.

Appendix A is provided as an aid to the reader. In Appendix A the definitions of inertial and body axes are developed. Transformations between inertial and body axes as well as diagrams illustrating their relationship to one another are shown.

The minimum set of terms in the equations of motion of a V/STOL aircraft that enable a description of its conventional mode of flight will be designated as a necessary description of the particular V/STOL aircraft. When the aircraft is in such a normal flight attitude, fewer axis systems and aerodynamic coefficients may be needed for description than during hover or transition flight. For some V/STOL configurations the axis systems available for fixed-wing aircraft will be sufficient; however for other V/STOL configurations additional axes must be defined. The capability of a V/STOL aircraft to change its physical configuration during vertical and transitional flight may indicate the need of additional axis systems. In addition, the consideration of available aerodynamic data for a particular vehicle may indicate the inclusion or exclusion of certain aerodynamic coefficients while traversing the V/STOL region. The additional terms generated by consideration of the V/STOL region, together with the minimum set of terms developed by consideration of the conventional flight regime, will constitute a necessary and sufficient description of the equations of motion for the V/STOL aircraft.

It is to be expected that any physical change in the actual configuration of the aircraft will probably necessitate the definition of additional axes, but the existence of a change in aircraft configuration or the use of additional thrust producers in the V/STOL region of flight does not necessarily force the development of new axis systems in addition to those already specified for the fixed-wing aircraft. Each type of aircraft must be treated individually and new axes developed only when the effects of the special features of the aircraft configuration cannot be accurately, economically and adequately handled with the axis systems used to describe the conventional mode of flight.

The most simple case of configuration change would be that causing physical rotation of the thrust vector. This can be accomplished by tilting one or more wings at the same or different rates or by tilting two or more ducted fans, propellers or rotors. Thrust rotation can also be accomplished by deflecting the flow from a thrust source such as a ducted fan. Yet another method of achieving V/STOL capability which may require special axis system treatment is the use of special thrust producers in the V/STOL region of flight that are completely separate from the thrust producers utilized in the conventional mode of flight. These separate sources of thrust take the form of jet engines, fans, or completely discrete paths for jet exhaust.

Possible axis systems for aircraft configurations involving such devices are presented below.

PARTICULAR V/STOL CONFIGURATIONS

We will now consider some examples of the use of additional axes to describe particular V/STOL aircraft. In the final consideration of a given V/STOL axis system the form of coefficient and performance data available for the given aircraft will contribute, to a certain extent, to the choice of axis systems for simulation of that aircraft.

TILT-WING. A current example of tilt-wing aircraft is the Tri-Service XC-142A Transport under development by Ling-Temco-Vought, Hiller Aircraft and Ryan Aeronautical Company. Figure 1 is a line drawing of this aircraft. The XC-142A is powered by four G.E. T-64 turbo-prop engines. These engines are mounted on a wing which can be tilted to an angle of 100 degrees from the x-y plane of the aircraft body axes such that hover in a tail wind is possible. The tail assembly is composed of a vertical fin and rudder, a movable horizontal stabilizer (no elevator) and a tail rotor for pitch stabilization.

In order to reduce the problem of keeping track of the relative orientation of the wing chord and the wind as well as the orientation of resulting force vectors, a wing axis system is defined. This system has its origin at the wing pivot point and has its x-z plane coincident with the body axis x-z plane. The system rotates with the wing such that the wing x axis is always parallel to the wing chord. Further, a set of wing stability axes is defined analogous to stability axes of fixed-wing aircraft. This system is rotated about the wing y axis by the angle of attack of the wing; i.e., the angle formed by the projection of the aircraft velocity vector on the x-z plane of the wing axis system and the wing x axis.

Also, in the transition and hover regions the forces developed at each propeller become the primary components of the stability of the aircraft. Therefore, an axis system (the propeller axis system) is specified for each main propeller. It may also be necessary to develop a propeller axis system for the tail rotor of some similar configurations, but this is not necessary for the XC-142A as shown in subsequent portions of this report.

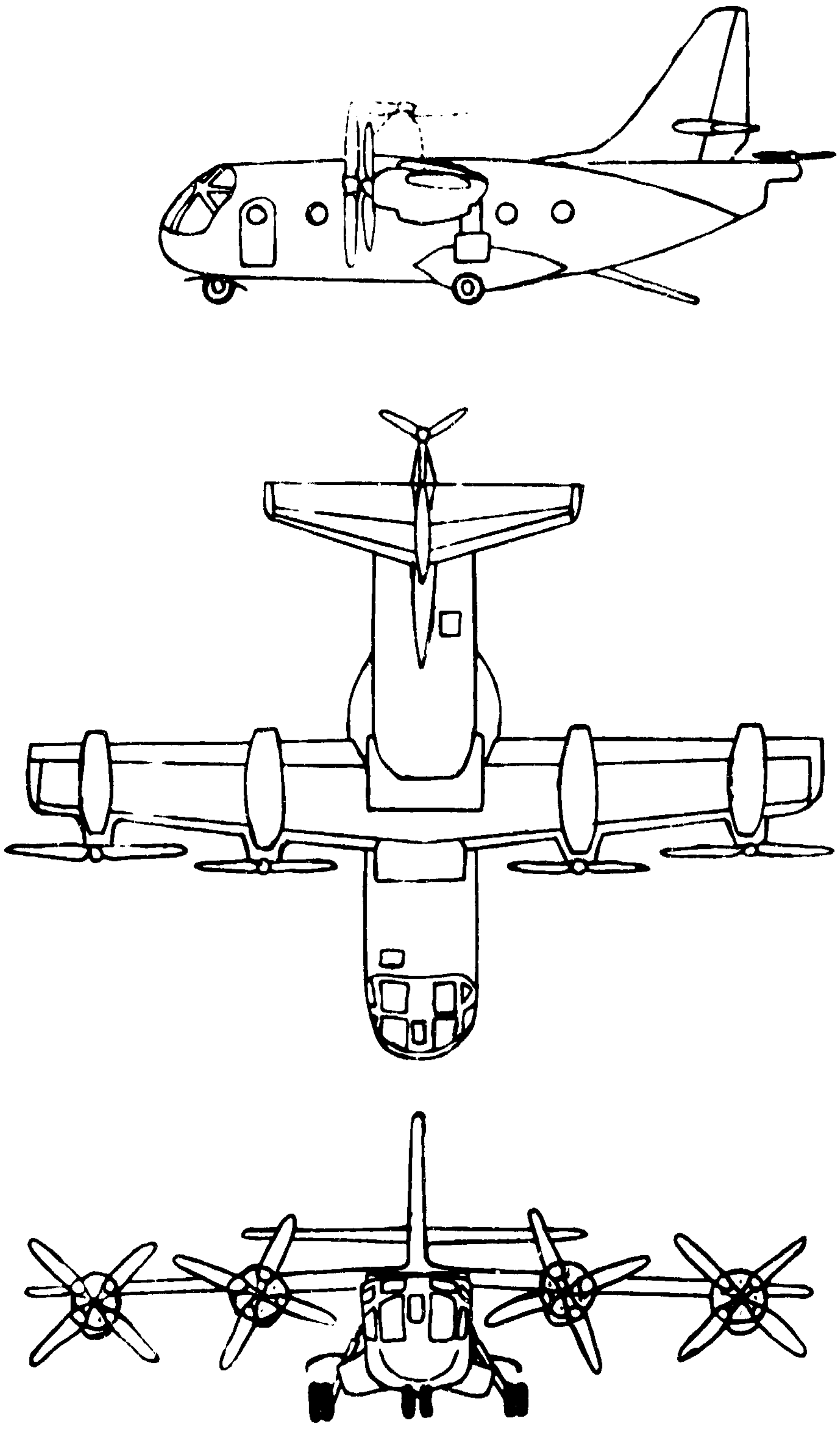


Figure 1. Three-View of XC-142A

These axis systems enable the development of force and moment expressions for the wing, main propellers and the tail rotor. The resulting forces and moments are transformed to aircraft body axes and there incorporated into the total force and moment equations. These axes in conjunction with inertial, body and wind axes would comprise a sufficient description of axes for simulation of a tilt-wing transport.

TILT-DUCT. An example of the tilt-duct is the VZ-4DA of DOAK Aircraft Company. This is a small aircraft with tiltable ducted fans, one at each wing tip. The tail assembly is composed of a vertical fin and rudder, a horizontal stabilizer and reaction controls. As for the tilt-wing it is desirable to compute the direction and magnitude of the thrust vectors from each ducted fan in order to simulate a trim aircraft in the V/STOL region. In order to accomplish this each rotatable ducted fan should contain an axis system. The resulting force and moments are transformed to the aircraft body axes by considering the angle of tilt of the ducts with respect to the wing. Figure 2 is a line drawing of this aircraft.

Another example of the tilt-duct is the proposed 4 ducted fan VTOL transport, the Bell D2127 of Bell Aerosystems Company. Here again axes for duct velocities and duct thrust vectors are desirable. Attached to the aft side of each duct are wing areas in line with the air flow. These wing areas rotate with the ducts. Forces and moments created by these surfaces can be computed in the axis system defined for the ducts, and then transformed to fuselage axes through the duct tilt angle, thus enabling summation of total forces and moments. In general, the thrust output of a ducted fan varies with its direction of flight with respect to axes located in the duct. The thrust output is not necessarily collinear with the axis of the fan. The components of force perpendicular to this axis become significant in the total moment equations and in general, these components of force have varying lever arms on which to act and become particularly significant terms in vertical or near vertical flight.

TILT-PROP. An example of this type of aircraft is the Curtiss-Wright X-19. Figure 3 is a line drawing of this aircraft which has two engines driving four propellers. Four tilting propellers are mounted in nacelles at the tips of tandem high wings. Stabilization results from variance in the control of pitch at each propeller. Pitch angle control comes from simultaneous change in the pitch of the forward propellers and the aft propellers respectively. Roll angle control results from simultaneous change in the pitch of starboard propellers and the port propellers respectively. Yaw angle control is developed from simultaneous change in the propeller pitch of diagonally opposite propellers. Since the propeller nacelles rotate, a nacelle incidence angle is defined with respect to the aircraft body axes. In order to account for the variance in magnitude and direction of the forces developed at each propeller individual axis systems will be defined. The propeller axis systems in addition to the axis systems of the conventional aircraft enable a sufficient description of the equations of motion of the X-19. Propeller axes are required because of the importance of nonaxial components of thrust in each nacelle.

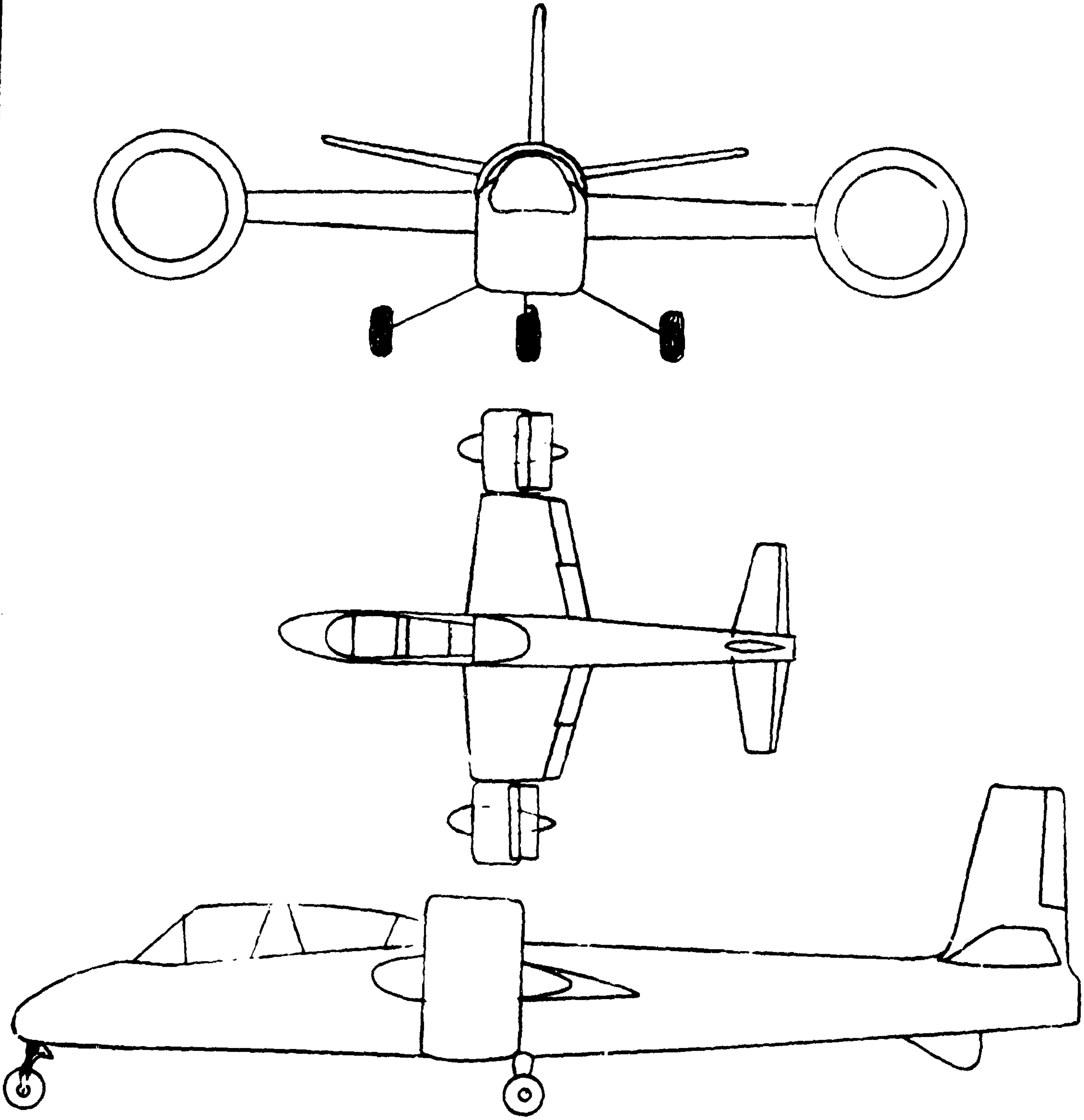


Figure 2. Three-View of VZ-4DA.

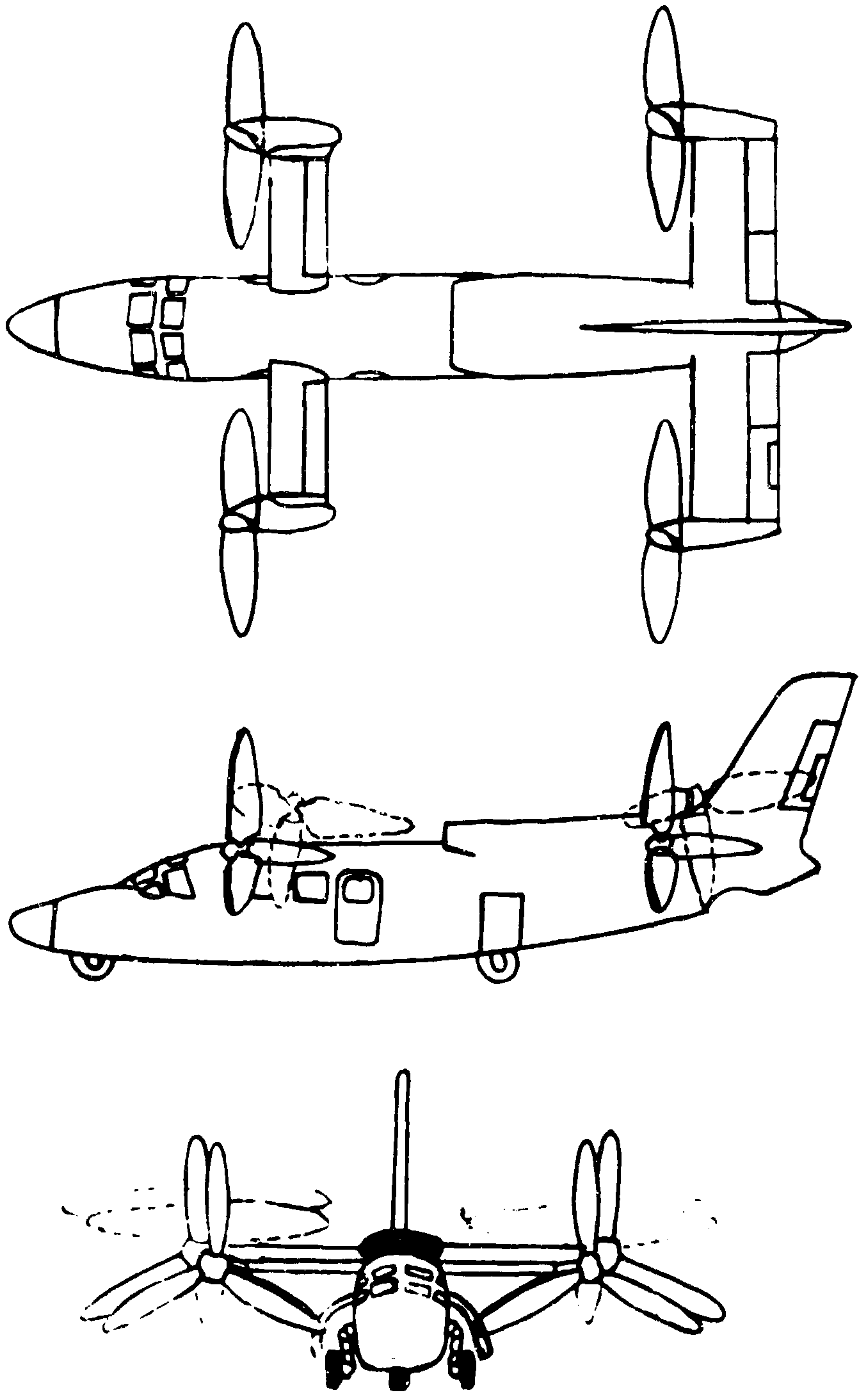


Figure 3. Three-View of X-19.

FAN-IN-WING. An example of this type of aircraft is the General Electric/ Ryan Aeronautical XV-5A V/STOL research aircraft. Figure 4 is a line drawing of such an aircraft. For this aircraft exhaust gases are emitted straight out the tail pipes in normal flight. For transition to, and operation in, vertical flight the tailpipes are blocked off and the exhaust gases are diverted to three free wheeling fans--one in the nose and a pair of large lift fans embedded in the wings. The thrust developed out of each fan is controlled by fan exit louvers. The thrust is proportional to fan rpm, and its direction is controlled by the setting of the louvers. If thrust versus fan rpm data and louver position versus thrust data were available such that the direction and magnitude of the thrust at each fan could be described, then forces and moments could be developed in existing wind and body axis systems so that the aircraft could be simulated by methods already developed for subsonic jet aircraft. However, if necessary, an axis system describing louver position could be developed in order to keep track of fan thrust during transition and hover conditions.

ROTATING THRUST AND DEFLECTED THRUST. An example of rotating thrust is the Hawker P.1127 transonic V/STOL strike fighter. Figure 5 is a line drawing of such an aircraft. A deflected thrust aircraft example is the VTOL trainer support aircraft, X-14A, by Bell Aerosystems Company. The P.1127 is powered by the Bristol Siddeley BS 53 engine which has four simultaneous rotating nozzles. The thrust being ejected from each nozzle is capable of being rotated through an angle of 100 degrees from a positive X-aircraft body axis. Nozzle deflection angles over 90 degrees allow for braking flight during a STOL maneuver. For this system we keep track of the thrust vectors from the exhaust nozzles of the powerplant. This can be done by tracking the rotation of the main exhaust nozzles with respect to body axes. No extra axis systems appear to be necessary.

In the case of the X-14A, thrust diverter position and power output of the engines may give enough information to determine thrust magnitude and orientation.

For both the P.1127 and X14-A, no additional axis systems are necessary for the development of the equations of motion for the purpose of simulation. Aerodynamic data should enable thrust calculations in wind or body axes, and the consequent direct computation of aerodynamic forces and moments.

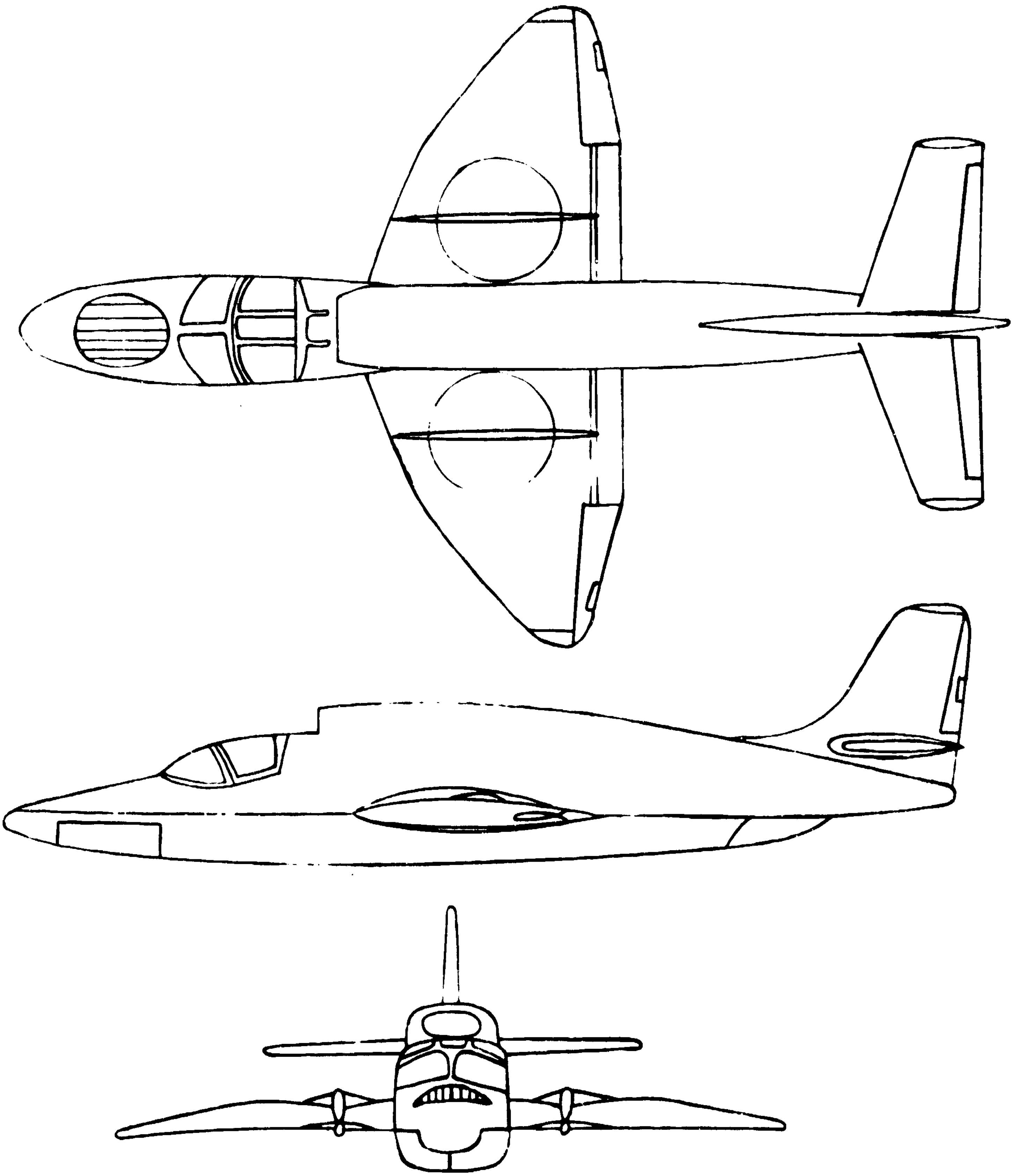


Figure 4. Three-View of XV-5A.

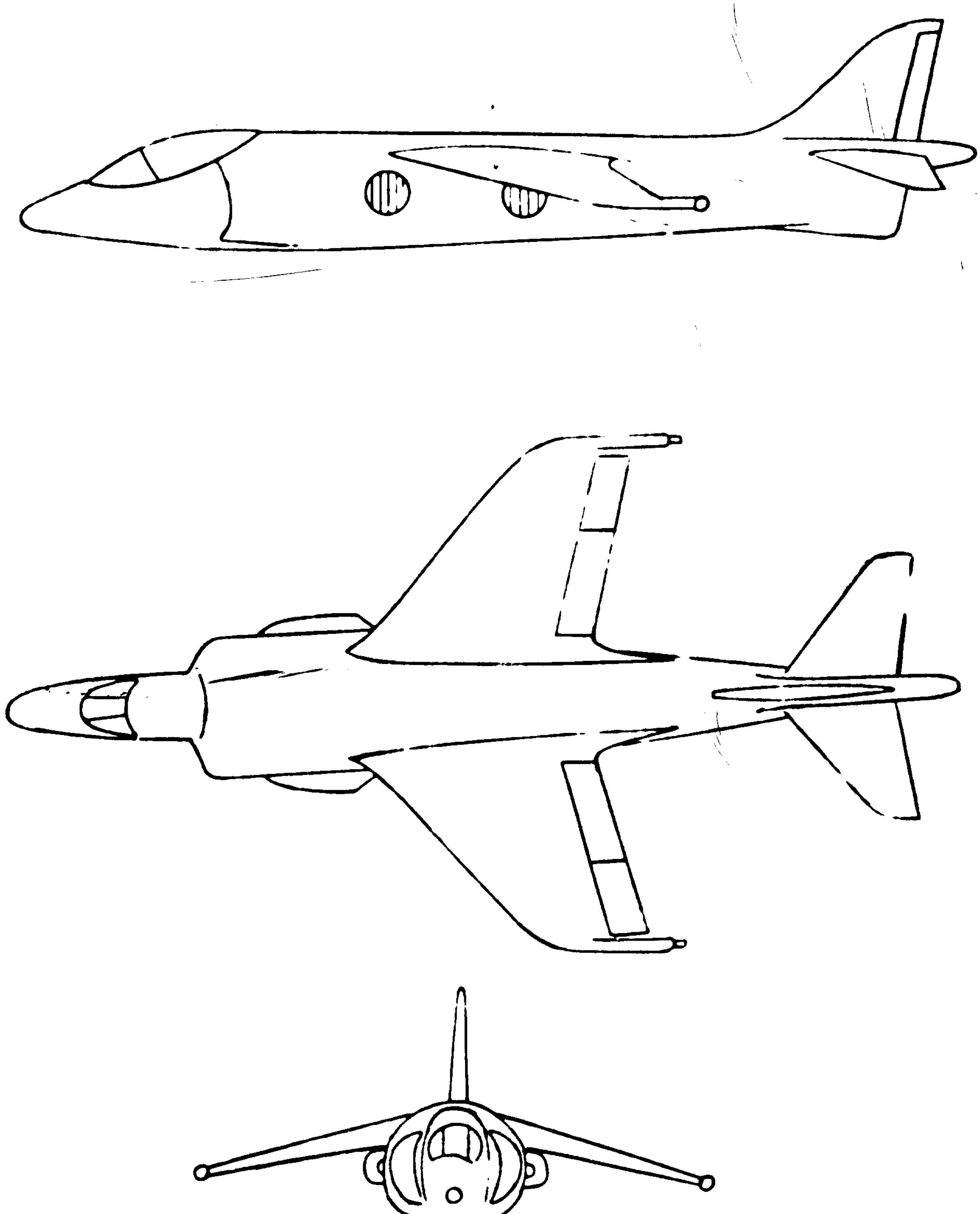


Figure 5. Three-View of P.1127.

SECTION III

MATHEMATICAL MODEL

In this section we will construct a mathematical model which can be applied to the simulation of any V/STOL aircraft. With this model, equations of motion for various V/STOL aircraft configurations are evolved in Section IV. The approach used in deriving these equations of motion will enable the reader to develop simulation equations--as needed--in a systematic and simple manner for V/STOL configurations that are evolved in the future.

The method of attack is to start with the equations of motion representing the conventional mode of operation of the V/STOL aircraft; to choose appropriate axis systems based upon the particular aircraft which are in addition to the inertial, body, stability and wind axes; to determine the V/STOL aerodynamic characteristics not simulated by the aircraft in its conventional flight mode; and finally to write the equations of motion for the V/STOL aircraft. The first objective is to develop the idea of a conventional flight mode.

CONVENTIONAL FLIGHT MODE OF V/STOL AIRCRAFT

The V/STOL aircraft (in the conventional mode of flight) is described by a set of necessary equations. There is no attempt to define whether they apply to single or multi-engine propeller or jet aircraft. The configuration of the aircraft is not defined other than there are control inputs for rudder, elevator, flap and aileron. There is no indication as to low or high performance or mission. Modifications of the basic simulation equations describing this model will depend upon the particular V/STOL aircraft. Consequently, specifying the particular aircraft enables the addition, if warranted, of the factors which allow the necessary equations of the aircraft operating in the conventional flight mode to become a necessary and sufficient set of simulation equations. In conjunction with the following, the development in Appendix A, and Reference 2, we are able to write the equations for a V/STOL aircraft in the conventional flight mode.

MATHEMATICAL MODEL OF CONVENTIONAL FLIGHT MODE OF V/STOL AIRCRAFT. The conventional flight mode is represented by a set of three force and three torque equations. Linear and angular rates are obtained from these equations.

Linear Motion. The following are the force expressions[(1)(2)].

$$F_e^{Ni} = F_I^{Ni} \qquad (i = 1, 2, 3)$$

e = external
I = internal
N = inertial axes

(1) See Appendix B of NAVTRADEVCEN 1205-1 and Appendix A herein.
(2) Connelly, reference 2, develops similar equations of motion.

The $\sum F_e^{Ni}$ are the total external forces which are composed of aerodynamic and gravity forces and the $\sum F_I^{Ni}$ are the total internal forces. We can view the aircraft as a system of mass points such that the distance between pairs of mass points are constant throughout the aircraft's motion. This is a rigid body, and consequently work done by internal forces between pairs of mass points vanishes. However, since we have a rigid body the center of mass of the mass points (m_I) is concentrated at a single point and the radius vectors (r_I) from the inertial frame to each arbitrary mass point are replaced by a radius vector r to the center of mass. We then have the following:

$$\sum F_e^{NI} = \sum m_I \ddot{r}_I^{Ni} = M\,\ddot{r}^{Ni} \tag{3.1}$$

$$M = \sum m_I \text{ mass of the aircraft}$$

Equation (3.1) is expanded to give:

$$\begin{aligned} \sum F_e^{N1} &= M\,\ddot{r}^{N1} \\ \sum F_e^{N2} &= M\,\ddot{r}^{N2} \\ \sum F_e^{N3} &= M\,\ddot{r}^{N3} \end{aligned} \tag{3.2}$$

To facilitate the description of forces aircraft body axes are located at the center of mass of the aircraft. Forces described in the N_i axes are transformed to the X_i body axes as shown in Appendix A. We have then the following equations:

$$\begin{aligned} \sum F_e^{X1} &= M(\dot{U} + Wq_1 - V\,r) \\ \sum F_e^{X2} &= M(\dot{V} + U\,r - W\,p) \\ \sum F_e^{X3} &= M(\dot{W} + V\,p - U\,q_1) \end{aligned} \tag{3.3}$$

In equations (3.3) the following identifications are made:

$$\sum F_e^{Xi} = \text{Aerodynamic Forces} + \text{Gravity Force} + \text{Engine Thrust}$$

where X1 = x, X2 = y, X3 = z, T = engine thrust, and α_T is the angle of the thrust line with respect to the x-body axis.

So that there is

$$\sum F_e^x = X_a - M\,g\,\sin\theta + T\cos\alpha_T$$

$$\sum F_e^y = Y_a + M\,g\,\cos\theta\,\sin\Phi \qquad (3.4)$$

$$\sum F_e^z = Z_a + M\,g\,\cos\theta\,\cos\Phi - T\sin\alpha_T$$

From (3.3) and (3.4) the force equations in aircraft body axes are:

$$X_a = M\,(\dot{U} + Wq_1 - V\,r) + M\,g\,\sin\theta - T\cos\alpha_T$$

$$Y_a = M\,(\dot{V} + U\,r - W\,p) - M\,g\,\cos\theta\,\sin\Phi \qquad (3.5)$$

$$Z_a = M\,(\dot{W} + V\,p - Uq_1) - M\,g\,\cos\theta\,\cos\Phi + T\sin\alpha_T$$

Angular Motion. Following are the torque expressions for the aircraft in conventional flight. It is practically a universal practice to develop aircraft angular motion in body axes both in the aircraft manufacturing and simulator industry. For simulation purposes this practice avoids the added problem of transformations of the products and moments of inertia which would be required if other axis systems were used. From Appendix A we write for the torque components the usual dynamic terms involving products and moments of inertia, angular velocities and accelerations, and the coupling of angular velocities.

$$L_a = I_{11}\,\dot{p} - I_{13}\,(\dot{r} + pq_1) + (I_{33} - I_{22})\,q_1 r$$

$$M_a = I_{22}\,\dot{q}_1 - I_{13}\,(r^2 - p^2) + (I_{11} - I_{33})\,pr \qquad (3.6)$$

$$N_a = I_{33}\,\dot{r} - I_{13}\,(\dot{p} - q_1 r) + (I_{22} - I_{11})\,pq_1$$

In the equations of 3.6, the x-z plane of the aircraft body axes is a plane of symmetry. Consequently products of inertia (I_{12}, I_{21}, I_{32}) are zero. L_a, M_a and N_a are the aerodynamic coefficients necessary to describe any V/STOL aircraft in conventional flight.

	COEFFICIENT	FORCE OR TORQUE EXPRESSION
X_a	$C_x\ (\alpha,\ Ma)$	$\frac{1}{2}\ \rho\ V^2SC_x\ (\alpha,\ Ma)$
	$C_x(\beta)$	$\frac{1}{2}\ \rho\ V^2SC_x(\beta)$
	$C_{x_{\delta F}}$	$\frac{1}{2}\ \rho\ V^2SC_{x_{\delta F}}\ \cdot\ \delta F$
Y_a	C_{y_β}	$\frac{1}{2}\ \rho\ V^2SC_{y_\beta}\ \beta$
Z_a	$C_z\ (\alpha,\ Ma)$	$\frac{1}{2}\ \rho\ V^2SC_z\ (\alpha,\ Ma)$
	$C_{z_{\delta F}}$	$\frac{1}{2}\ \rho\ V^2SC_{z_{\delta F}}\ \cdot\ \delta F$
	$C_{z_{\delta E}}$	$\frac{1}{2}\ \rho\ V^2SC\ C_{m_{\delta E}}\ \cdot\ \delta E$
L_a	C_{l_β}	$\frac{1}{2}\ \rho\ V^2Sb\ C_{l_\beta}\ \cdot\ \beta$
	$C_{l_{\delta A}}$	$\frac{1}{2}\ \rho\ V^2Sb\ C_{l_{\delta A}}\ \cdot\ \delta A$
	$C_{l_{\delta R}}$	$\frac{1}{2}\ \rho\ V^2\ Sb\ C_{l_{\delta R}}\ \cdot\ \delta R$
	$C_{l_{\delta R}}$	$\frac{1}{2}\ \rho\ V^2\ Sb\ C_{l_{\delta R}}\ \cdot\ \delta R$
	C_{l_p}	$\frac{1}{4}\ \rho\ VSb^2\ C_{l_p}\ \cdot\ p$
	C_{l_r}	$\frac{1}{4}\ \rho\ VSb^2\ C_{l_r}\ \cdot\ r$

Table 1. Dimensionless Coefficients Used for Basic Aircraft

	COEFFICIENTS	FORCE OR TORQUE EXPRESSION
M_a	C_m (α, Ma)	$\frac{1}{2} \rho V^2 SC\, C_m$ (a, Ma)
	$C_{m_{\delta F}}$	$\frac{1}{2} \rho V^2 Sc\, C_{m_{\delta F}} \cdot \delta F$
	$C_{m_{\delta E}}$	$\frac{1}{2} \rho V^2 Sc\, C_{m_{\delta E}} \cdot \delta E$
	$C_{m_{q_1}}$	$\frac{1}{4} \rho V Sc^2\, C_{m_{q_1}} \cdot q_1$
	$C_{m_{\dot{\alpha}}}$	$\frac{1}{4} \rho V Sc^2\, C_{m_{\dot{\alpha}}} \cdot \dot{\alpha}$
N_a	C_{n_β}	$\frac{1}{2} \rho V^2 Sb\, C_{n_\beta} \cdot \beta$
	$C_{n_{\delta A}}$	$\frac{1}{2} \rho V^2 Sb\, C_{n_{\delta A}} \cdot \delta A$
	$C_{n_{\delta R}}$	$\frac{1}{2} \rho V^2 Sb\, C_{n_{\delta R}} \cdot \delta R$
	C_{n_p}	$\frac{1}{4} \rho V Sb^2\, C_{n_p} \cdot p$
	C_{n_r}	$\frac{1}{4} \rho V Sb^2\, C_{n_r} \cdot r$

$V \equiv$ Aircraft velocity

$\rho \equiv$ Air density

$S \equiv$ Wing area

$b \equiv$ Wing span

$c \equiv$ Mean aerodynamic chord

Table 1. Dimensionless Coefficients Used for Basic Aircraft (Cont'd.)

EXAMINATION OF COEFFICIENTS FOR V/STOL AIRCRAFT IN CONVENTIONAL FLIGHT MODE. Following the work presented by Connelly in Reference 2, we will state that the coefficients listed in Table 1 constitute a necessary number of aerodynamic coefficients in order to develop the aerodynamic forces and moments for a V/STOL aircraft in conventional flight. In many cases actual manufacturers' data may not be available for just these specific coefficients. For example, data are usually available in terms of coefficient of lift (C_L) or drag (C_L) which in turn can be transformed to give C_x and C_z. Table 1 is then a list of what is considered to be a minimum number of aerodynamic coefficients to describe the aircraft in conventional flight. Modification of this list is possible when the flight of the aircraft is described in the V/STOL region.

For an actual V/STOL aircraft the equations for X, Y, Z, L_a, M_a and N_a may contain terms due to the specific effect of various aircraft components--the wing, propeller(s) or ducted fan, the fuselage, vertical tail and/rudder, horizontal tail, and devices to insure trim flight such as a small variable direction thrust propeller or ducted fan. Each of these forces and moments due to the various aircraft components will have associated aerodynamic coefficients--some common to the aircraft in conventional flight and other aerodynamic coefficients applicable only to the particular V/STOL aircraft.

STATEMENT OF EQUATIONS FOR V/STOL AIRCRAFT IN CONVENTIONAL FLIGHT MODE. From the force equations (3.5) and the torque equations (3.6) and the data of Table 1 we can write a set of equations for the aircraft in conventional flight. It must be remembered that these equations are only guidelines. They give us an insight into what will be contained in the simulation of a V/STOL aircraft. The equations which follow (3.7 to 3.12) are written in aircraft body axes and do not include effects due to landing gear or other variations in external stores or the aircraft configuration.

X Force Equation

$$\frac{1}{2}\rho V^2S[C_x(\alpha,Ma) + C_x(\beta) + C_{x_{\delta F}}\delta F] =$$

$$m(\dot{U} + Wq_1 - Vr) + mg\sin\theta - T\cos\alpha_T \tag{3.7}$$

or $$(\dot{U} + Wq_1 - Vr) = \rho\frac{V^2S}{2m}[C_x(\alpha, Ma) + C_x(\beta) + C_{x_{\delta F}}\cdot\delta F] - g\sin\theta + \frac{T}{m}\cos\alpha_T$$

Y Force Equation

$$\frac{1}{2}\rho V^2S(y_\beta\beta = m(\dot{V} + Ur - Wp) - mg\cos\theta\sin\Phi$$

or $$(\dot{V} + Ur - Wp) = \rho\frac{V^2S}{2m}C_{y_\beta}\cdot\beta + g\cos\theta\sin\Phi \tag{3.8}$$

Z Force Equation

$$\frac{1}{2}\rho V^2 S\,[C_z(\alpha,\ Ma) + C_{z_{\delta F}} \cdot \delta F + C_{z_{\delta E}} \cdot \delta E] \qquad (3.9)$$

$$= m(\dot{W} + Vp - Uq_1) - mg \cos\theta \cos\Phi + T \sin\alpha_T$$

or

$$(\dot{W} + Vp - Uq_1) = \rho \frac{V^2 S}{2m}[C_z(\alpha,\ Ma) + C_{z_{\delta F}} \cdot \delta F + C_{z_{\delta E}} \cdot \delta E]$$

$$+ g \cos\theta \cos\Phi - \frac{T}{m}\sin\alpha_T$$

Roll Equation

$$\frac{1}{4}\rho\, VSb(2VC_{l_\beta} \cdot \beta + 2VC_{l_{\delta A}} \cdot \delta A + 2VC_{l_{\delta R}} \cdot \delta R + bC_{l_p} \cdot p \qquad (3.10)$$

$$+ bC_{l_r} \cdot r)$$

$$= I_{11}\dot{p} - I_{13}(\dot{r} + pq_1) + (I_{33} - I_{22})\, q_1 r$$

or

$$\dot{p} = + \frac{I_{13}}{I_{11}}(\dot{r} + pq_1) - \frac{(I_{33} - I_{22})}{I_{11}} q_1 r$$

$$+ \rho \frac{VSb}{4I_{11}}(2VC_{l_\beta} \cdot \beta + 2VC_{l_{\delta A}} \cdot \delta A + 2VC_{l_{\delta R}} \cdot \delta R + b\, C_{l_p} \cdot P + b\, C_{l_r} \cdot r)$$

Pitch Equation

$$\frac{1}{4}\rho\, VSc[2VC_m(\alpha,\ Ma) + 2VC_{m_{\delta F}} \cdot \delta F + 2VC_{m_{\delta E}} \cdot \delta E + c\, C_{mq_1} \cdot q_1] \qquad (3.11)$$

$$= I_{22}\dot{q}_1 - I_{13}(r^2 - p^2) + (I_{11} - I_{33})\, pr$$

or

$$\dot{q}_1 = + \frac{I_{13}}{I_{22}}(r^2 - p^2) - \frac{(I_{11} - I_{33})}{I_{22}}\, pr$$

$$+ \rho \frac{VSc}{4I_{22}}[2VC_m(\alpha, Ma) + 2VC_{m_{\delta F}} \cdot \delta F + 2VC_{m_{\delta E}} \cdot \delta E + cC_{m_{q_1}} \cdot q_1 - cC_{m_{\dot{\alpha}}} \cdot \dot{\alpha}]$$

Yaw Equation

$$\frac{1}{4}\,\rho\,VSb(2VC_{n_\beta}\cdot\beta + 2VC_{n_{\delta A}}\cdot\delta A + 2VC_{n_{\delta R}}\cdot\delta R + bC_{n_p}\cdot p + bC_{n_r}\cdot r) \tag{3.12}$$

$$= I_{33}\dot{r} - I_{13}(\dot{p} - q_1 r) + (I_{22} - I_{11})\,pq_1$$

or

$$\dot{r} = +\frac{I_{13}}{I_{33}}(\dot{p} - q_1 r) - \frac{(I_{22} - I_{11})}{I_{33}}\,pq_1$$

$$+\,\rho\,\frac{VSb}{4I_{33}}(2VC_{n_\beta}\cdot\beta + 2\,VC_{n_{\delta A}}\cdot\delta A + 2VC_{n_{\delta R}}\cdot\delta R + bC_{n_p}\cdot p + bC_{n_r}\cdot r)$$

ADDITIONAL AXIS SYSTEM SELECTION

The next step in the mathematical model is to choose the necessary additional axis systems in order to describe the particular flight performance of a given V/STOL aircraft. In Section II axis systems were discussed in conjunction with particular V/STOL configurations.

V/STOL AERODYNAMIC COEFFICIENTS

Once a system of axes is selected, additional coefficients may be specified. These could relate to thrust for jet, propeller of ducted fan. They could be functions for speed brakes, slats, external stores, landing gear, drop tanks, bomb bay doors or louvers for control of air flow. The inclusion of any such coefficients is best illustrated by the tilt-wing example in Section IV.

V/STOL EQUATIONS OF MOTION

With the equations of the V/STOL aircraft in conventional flight, axis systems, and a study of the particular V/STOL aerodynamics discussed up to this point, we are now ready to write the equations of motion for the V/STOL aircraft. No small angle, or small term simplifications have been made in the force or torque equations of the aircraft in conventional flight. Simplifications may be made during the development of the equations for a particular aircraft.

SECTION IV

EQUATIONS OF MOTION FOR V/STOL AIRCRAFT

In this section equations of motion will be developed for five V/STOL configurations. These configurations--tilt-wing, tilt-duct, tilt-prop, fan-in-wing and rotating thrust--incorporate individually and collectively the designs of V/STOL aircraft currently being developed. A particular aircraft, either one that is proposed or a prototype will be used as an example of each of these configurations. The first type to be considered is the tilt-wing.

TILT-WING

The Vought-Hiller-Ryan XC-142A Tri-Service Transport will be used in the example in the development of tilt-wing equations of motion. The equations from Section III will be modified so that they are applicable to an aircraft such as the XC-142A.

In order to obtain some idea of the XC-142A let us consider Figure 1, Table 2 and Table 3. In Figure 1, a three view arrangement of the XC-142A is shown. Table 2 contains some of the physical characteristics of the aircraft and Table 3 is a list of definitions of symbols contained in the XC-142A equations.

With this general idea of the XC-142A, we may now develop the mathematical model for the XC-142A. First, define, if necessary, applicable axis systems. Second, develop additional aerodynamic coefficients. Finally, write the equations of motion for the XC-142A.

CHARACTERISTIC	VALUE
Normal Gross Weight	37,474 pounds
Center of Gravity (aft of the leading edge of the mean aerodynamic chord (MAC))	
Max Forward	10% MAC
Max Aft	28% MAC
Wing	
Total Area	534.37 ft^2
Span	67.5 ft
Aspect Ratio	8.53
Dihedral Angle	-2.12°
Airfoil Section	NACA 63-318 (Mod)
Trail Edge Flaps - Double slotted	
Maximum Deflection	60°
Deflection for take off (STOL)	40°
Deflection for landing (STOL)	60°
Leading Edge Flaps	
Deflection	87°
Ailerons - Plain	
Maximum Deflection (wing up)	$\pm 50^\circ$
Maximum Deflection (wing down)	$\pm 15^\circ$
Horizontal Stabilizer - All moving	
Area	163.5 ft^2
Span	31.14 ft
Aspect Ratio	5.08
Dihedral Angle	0°
Airfoil Section (root)	NACA 0015
Airfoil Section (tip)	NACA 0012
Maximum Deflection (leading edge up)	60°
Maximum Deflection (leading edge down)	15°
Hinge Line, % of tail mean geometric chord	13%
Vertical Tail	
Area: Fin to rudder hinge	95 ft^2
Area: Rudder aft of hinge	21.6 ft^2
Aspect Ratio	1.87
Airfoil Section (root)	NACA 0018
Airfoil Section (tip)	NACA 0012
Fuselage	
Length	50 ft
Length (including tail rotor)	58.12 ft
Outside Height	10.72 ft
Outside Width	9.25 ft
Maximum cross-sectional area	90 ft^2
Propellers	
Diameter	15.5 ft
Number of Blades	4
Tail Rotor	
Diameter	8 ft
Number of Blades	3

Table 2. Selected Physical Characteristics XC-142A

Symbol	Definition
F	Fuselage
W	Wing
n	Main propellers n = 1, 2, 3, 4 - top view left to right
vt	Vertical Tail
hs	Horizontal Stabilizer
TR	Tail Rotor
α_F	Angle of attack - fuselage
α_w	Angle of attack - wing
α_{hs}	Angle of attack - horizontal stabilizer
i_t	Incidence angle - horizontal stabilizer
ϵ	Downwash angle
β_F	Sideslip angle - fuselage
β_w	Sideslip angle - wing
i_w	Incidence angle - wing
ξ	$\xi = i_w - \alpha_w$
V_B	Total velocity in aircraft body axes
V_w	Total velocity in wind stability axes
c	Mean aerodynamic chord
b	Wing span
S	Total area of wing
S_p	Total area of main propeller disks
q_w	Dynamic pressure due to power on wing effects
q	Dynamic pressure - free stream
q_{HS}	Dynamic pressure at horizontal stabilizer

Table 3. Definition of Symbols Used in XC-142A Equations

Symbol	Definition
$C_{T,S}$	Coefficient of thrust at wing due to inflow velocity
T	Total thrust of main propellers
J_n	Advance ratio main propeller
J_{TR}	Advance ratio tail rotor
N_n	RPM - main propeller
N_{TR}	RPM - tail rotor
N_o	Nominal RPM - main propeller
B_n	Blade pitch angle - main propeller
T_n	Main propeller thrust
N_n	Main propeller thrust component normal to T_n
Y_n	Main propeller moment (initially turning)
M_n	Main propeller moment (initially pitching)
Q_n	Main propeller torque
D_n	Diameter main propeller
D_{TR}	Diameter tail rotor
B_{TR}	Blade pitch angle - tail rotor
T_{TR}	Thrust - tail rotor
I_E	Inertia of main propeller blades and shaft
Ω_E	Angular velocity main propeller
$\dot{\Omega}_E$	Angular acceleration main propeller
I_{TR}	Inertia of tail rotor blades and shaft
Ω_{TR}	Angular velocity tail rotor
Ω_{TR}	Angular acceleration tail rotor

Table 3. Definition of Symbols Used in XC-142A Equations (Con'td.)

AXIS SYSTEMS FOR XC-142A. In order to describe the dynamics and aerodynamics of the tilt-wing aircraft, one axis system--the propeller axis system--in addition to the inertial, body, stability and wind axes is required. The wind axis system is complicated by the tilt of the wing. Figure 6 shows the wing tilt angle (i_w) and the variable distances ($x_{a.c.}$ and $z_{a.c.}$) that track the aerodynamic center (a.c.) of the wing as the wing is tilted through the angle i_w. The aerodynamic center of the wing as located in the x-z body axes plane is the origin of the wing stability axes. The wing stability axis system is parallel to the aircraft stability axis system and has its origin translated by $x_{a.c.}$ and $z_{a.c.}$ from the aircraft stability axis origin (which is located at aircraft c.g. nominally) as in Figure 6.

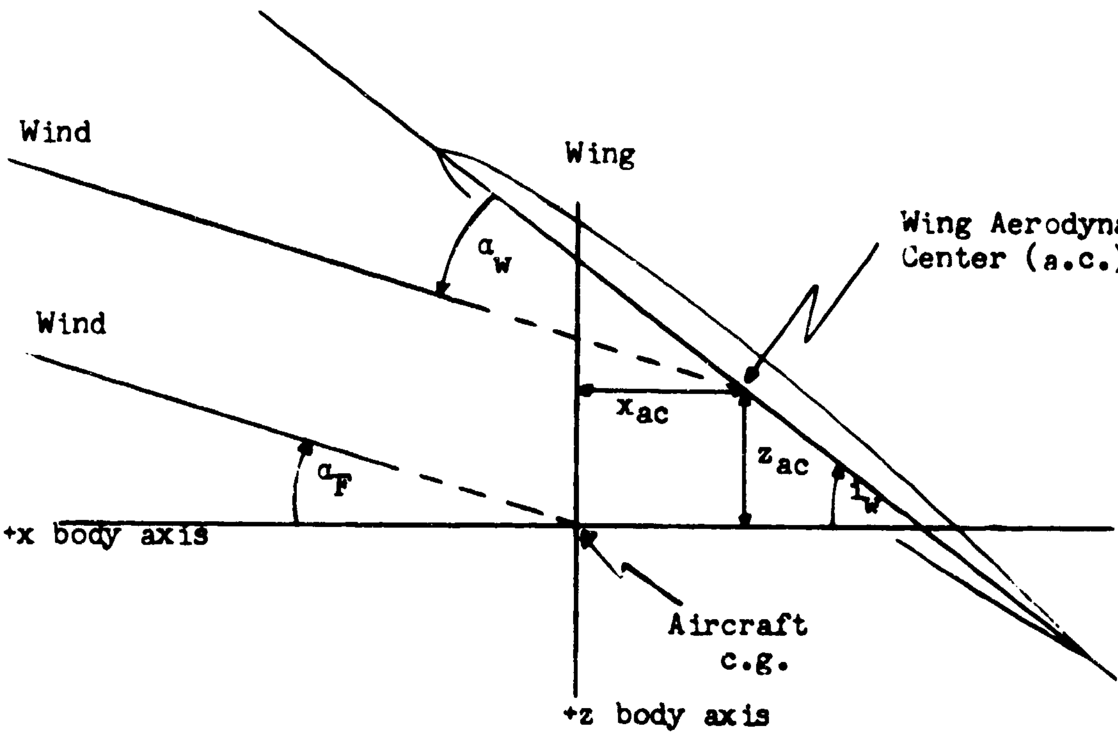
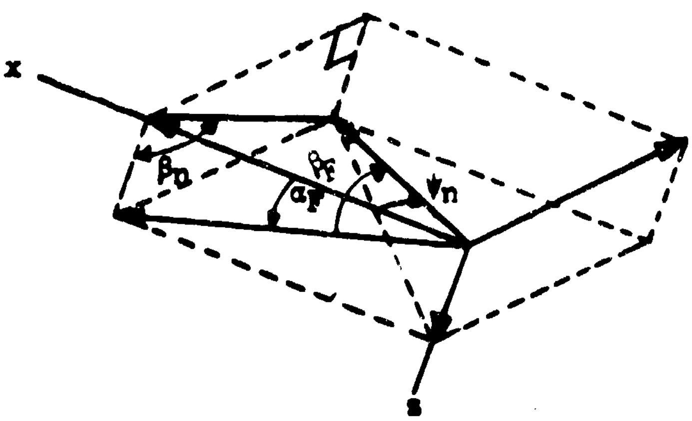


Figure 6. Wing Stability Axis System

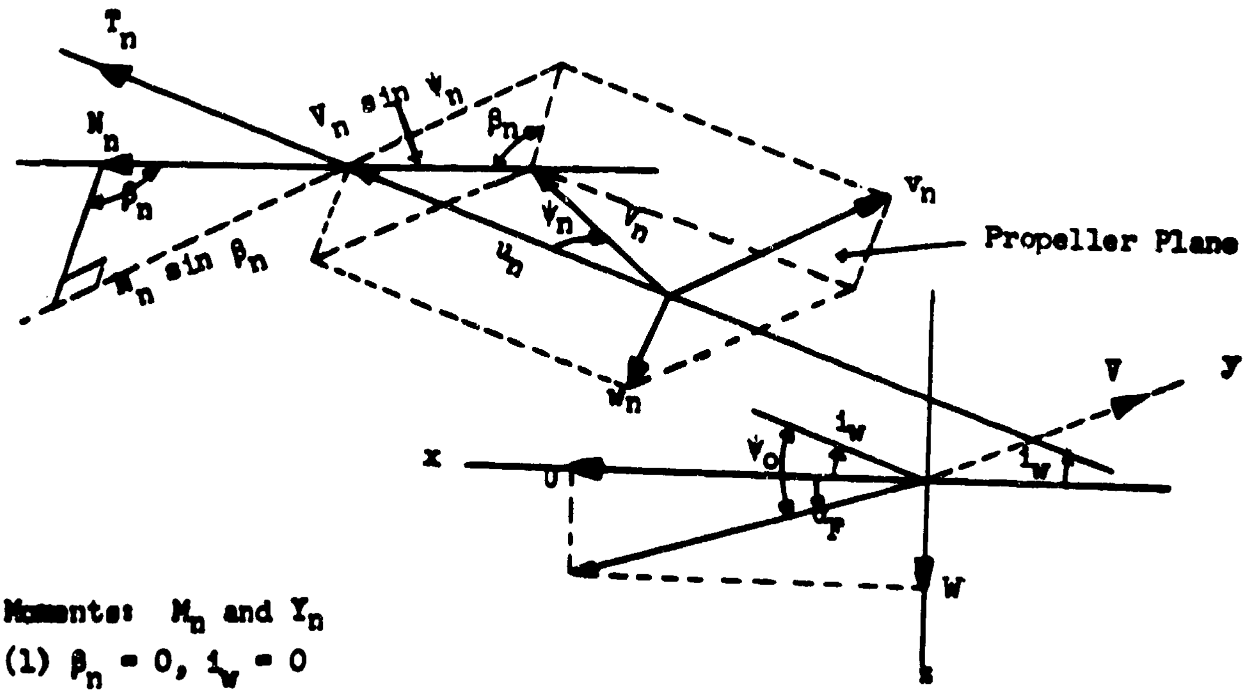
The additional axis system, the propeller axis system, is shown in Figure 7. The axis system is repeated for each main propeller, so that there are actually four main propeller axis systems. The propeller axis system will enable the development of the propeller thrust (T_n) propeller torque (Q_n) in terms of propeller power, a normal force (N_n) perpendicular to T_n along the propeller blade, and propeller moments (M_n) and (Y_n).

Further development of these propeller forces and moments will be considered in the discussion of aerodynamic effects.



$$\cos \psi_n = \cos \beta_F \cos \alpha_F$$

$$\sin \psi_n = \frac{\cos \beta_F \sin \alpha_F}{\cos \beta_n}$$



Moments: M_n and Y_n

(1) $\beta_n = 0,\ i_w = 0$
M_n is in the x-z plane
Y_n is in the x-y plane

(2) $\beta_n \neq 0,\ i_w = 0$
M_n tilted from x-z plane by $\measuredangle\ \beta_n$.
Y_n tilted from x-y plane by $\measuredangle (90-\beta_n)$
Obtain components of M_n and N_n
by multiplying by $\sin \beta_n$ and $\cos \beta_n$.

(3) $\beta_n \neq 0,\ i_w \neq 0$
In addition to (2) M_n rotated in x-z plane by $\measuredangle\ i_w$.
N_n rotated from x-y plane by $\measuredangle\ i_w$.

Figure 7. Main Propeller Axis System

At this point in the discussion we will recall the equations of motion of the basic aircraft as discussed in Section III. The internal moments (the right side of the moment equations) now include gyroscopic effects due to the rotating mass of an engine. The gyroscopic effects are due to the main engines and the tail rotor. For the main engines we have the following terms representing gyroscopic effects:

$+ (I_E \dot{\Omega}_E) \cos i_w - q_1 (I_E \Omega_E) \sin i_w$ for L_a term.

$+ p(I_E \Omega_E) \sin i_w + r(I_E \Omega_E) \cos i_w$ for M_a term.

$- (I_E \dot{\Omega}_E) \sin i_w - q_1 (I_E \Omega_E) \cos i_w$ for N_a term.

For the tail rotor we have

$+ q_1 I_{TR} \Omega_{TR}$ for L_a term.

0 for M_a term.

$- p I_{TR} \Omega_{TR}$ for N_a term.

In these terms p, q_1 and r are respectively the angular velocities around the x, y and z body axes. The wing tilt angle is i_w. I_E and I_{TR} are respectively, main engine and tail rotor inertias and Ω_E and Ω_{TR} are respectively, engine and tail rotor rotational speed.

The equations of motion without expansion of the aerodynamic terms and incorporating the engine gyroscopic and tail rotor terms become:

$$X_a = m(\dot{U} + Wq_1 - Vr) + mg \sin \theta$$

$$Y_a = m(\dot{V} + Ur - Vp) - mg \cos \theta \sin \Phi$$

$$Z_a = m(\dot{W} + Vp - Uq_1) - mg \cos \theta \cos \theta$$

$$L_a = I_{11}\dot{p} - I_{13}(\dot{r} + pq_1) + (I_{33} - I_{22})q_1 r + (I_E \dot{\Omega}_E) \cos i_w - q_1 (I_E \Omega_E) \sin i_w + q_1 I_{TR} \Omega_{TR}$$

$$M_a = I_{22}\dot{q}_1 - I_{13}(r^2 - p^2) + (I_{11} - I_{33})pr + p(I_E \Omega_E) \sin i_w + r(I_E \Omega_E) \cos i_w$$

$$N_a = I_{33}\dot{r} - I_{13}(\dot{p} - q_1 r) + (I_{22} - I_{11})pq_1 - (I_E \dot{\Omega}_E) \sin i_w - q_1 (I_E \Omega_E) \cos i_w - pI_{TR} \Omega_{TR}$$

The aerodynamic forces and moments--X_a, Y_a, Z_a, L_a, M_a and N_a will be developed for the tilt-wing aircraft so that a complete set of equations of motion is presented.

AERODYNAMIC FORCES AND MOMENTS - XC-142A. In the equations of motion developed for the basic aircraft in Section III a set of aerodynamic force and moment terms were developed as a function of specific dimensionless and aerodynamic coefficients. We will develop aerodynamic force and moment expressions based upon the particular configurations of the XC-142A tilt-wing aircraft.

In order to develop expressions for X_a, Y_a, Z_a, L_a, M_a and N_a, contributions from the major airframe components of the XC-142A aircraft will be considered separately. The major components to be considered are the wing, the main propellers, the vertical tail and rudder, the horizontal stabilizer, the tail rotor and the fuselage. After the aerodynamic force and moment expressions for each of these major components are developed, they will be summed to get the total aerodynamic force and moment expressions. These total force and moment expressions can then be visually checked with the aerodynamic force and moment expressions developed for the basic aircraft in Section III. The expressions in Section III will be contained in the aerodynamic force and moment expressions developed for the tilt-wing XC-142A.

Before continuing we will develop the aircraft body axes to inertial axes transformation. The matrices are:

$$\begin{bmatrix} N1 \\ N2 \\ N3 \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\Phi - \sin\psi\cos\Phi & \cos\psi\sin\theta\cos\Phi + \sin\psi\sin\theta \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\Phi + \cos\psi\cos\Phi & \sin\psi\sin\theta\cos\Phi - \cos\psi\sin\Phi \\ -\sin\theta & \cos\theta\sin\Phi & \cos\theta\cos\Phi \end{bmatrix} \times \begin{bmatrix} X1 \\ X2 \\ X3 \end{bmatrix}$$

In these matrices N1 is north, N2 is east and N3 down for the inertial axes, and X1 is x, X2 is y and X3 is z for the aircraft body axes.

ψ, θ and Φ are conventional Euler angles. Their definitions in developing the above matrices are shown in Appendix A.

The inertial to body axes rates are:

$$p = -\dot{\psi}\sin\theta + \dot{\Phi}$$

$$q_1 = \dot{\psi}\cos\theta\sin\Phi + \dot{\theta}\cos\Phi$$

$$r = \dot{\psi}\cos\theta\cos\Phi - \theta\sin\Phi$$

And conversely:

$$\dot{\Phi} = p + q_1 \tan\theta \sin\Phi + r \tan\theta \cos\Phi$$

$$\dot{\theta} = q_1 \cos\Phi - r \sin\Phi$$

$$\dot{\psi} = r \frac{\cos\Phi}{\cos\theta} + q_1 \frac{\sin\Phi}{\cos\theta}$$

We are now ready to develop the aerodynamic forces and moments for each major component of the aircraft. The first to be considered will be the effect of the main propellers. It should be noted that the expressions developed are for hovering, transition and conventional flight.

Main Propeller. There are four mutually similar systems of axes used to describe forces and moments generated by the main propellers during hover and low aircraft velocities. An analysis which was developed by Ling-Temco-Vought (Reference 6) for the XC-142A is adopted herein for the description of main propeller forces and moments. The subscript n, (n = 1, 2, 3, 4), denotes the particular propeller. They are numbered left to right looking from the top--1 and 2 are port propellers; 3 and 4 are starboard propellers. The approach used to develop the required propeller equations is to first consider the propeller geometry, next state the aerodynamic coefficients and finally write expressions for the force and moment contributions of the main propellers.

1. Main Propeller Geometry. From Figure 7 the wind vector with respect to each propellor is formed as $V_n^2 = u_n^2 + v_n^2 + w_n^2$.

For each propeller we may define an angle ψ_o which is the angle between the component of the total aircraft velocity (V_B) in the x-z plane of the body axes and the propeller axis line. If $i_w = 0$, then $\psi_o = \alpha_F$.

$$\psi_o = (i_w + \alpha_F)$$

From geometric considerations in Figure 7 we may form the expression for ψ_n which is the angle between the u_n and V_n velocities.

$$\cos\psi_n = \frac{u_n}{V_n} = \cos\beta_F \cos\psi_o \qquad \beta_F = \sin^{-1}\left(\frac{V}{V_B}\right)$$

$$\sin\psi_n = \frac{(w_n^2 + v_n^2)^{1/2}}{V_n} = \frac{\cos\beta_F \sin\psi_o}{\cos\beta_n} \qquad (4.1)$$

In a like manner expressions are formed for β_n which is defined as the angle between the velocities $V_n \sin \psi_n$ and w_n.

$$\sin \beta_n = \frac{V_n}{V_n \sin \psi_n} = \frac{\sin \beta_F}{\sin \psi_n}$$

$$\cos \beta_n = \frac{W_n}{(w_n^2 + v_n^2)\ 1/2} = \frac{\cos \beta_F \sin \psi_o}{\sin \psi_n} \tag{4.2}$$

The velocity expressions u_n, v_n and w_n will now be developed. The origins of each propeller axis are located along a line parallel to the y-body axis at approximately the center of mass of each nacelle. In Figure 7 the origins of the propeller axis systems are rotated from $V_B \cos \beta$ by angle ψ_o. Each propeller axis system origin is located by x_n, y_n and z_n body axis components which are multiplied by the appropriate body axis angular velocity in order to give tangential velocity components of u_n, v_n and w_n.

$$u_n = V_B \cos \beta_F \cos \psi_o - y_n(p \sin i_w + r \cos i_w) + x_n q_1 \sin i_w + z_n q_1 \cos i_w \tag{4.3}$$

$$v_n = V_B \sin \beta_F \tag{4.4}$$

$$w_n = V_B \cos \beta_F \sin \psi_o + y_n(p \cos i_w - r \sin i_w) - x_n q_1 \cos i_w + z_n q_1 \sin i_w \tag{4.5}$$

In equations (4.3) and (4.5) the term y_n is a constant. In fact $y_1 = y_4$ and $y_2 = y_3$ since the inboard and the outboard propellers are each the same distance from the x-z body axis plane. The component x_n and z_n vary as a function of wing tilt.

2. Main Propeller Aerodynamic Coefficients. The aerodynamic coefficients presented for the main propellers follow those presented by Reference 6. First let us define the advance ratio (J_n) for each main propeller and the advance ratio normal to the propeller disk (J_n').

$$J_n = \frac{60V_n}{N_n D} \text{ and } J_n' = J_n \cos \psi_n \tag{4.6}$$

The symbol N_n is the particular propeller RPM, the number 60 changes RPM to RPS, D is the diameter of the propeller and B_n the blade pitch angle of the particular propeller. The aerodynamic coefficients are then developed in terms of advance ratio and blade pitch.

$$C_{T_n} = C_{T_o} + \frac{\partial C_T}{\partial J'} J'_n + \frac{\partial^2 C_T}{\partial J'^2} (J'_n)^2 + \frac{\partial C_T}{\partial B} B_n + \frac{\partial^2 C_T}{\partial B \partial J'} B_n J'_n \qquad (4.7)$$

$$C_{p_n} = C_{P_o} + \frac{\partial C_p}{\partial J'} J'_n + \frac{\partial^2 C_p}{\partial J'^2} (J'_n)^2 + \frac{\partial C_p}{\partial B} B_n + \frac{\partial^2 C_p}{\partial B \partial J'} B_n J'_n \qquad (4.8)$$

$$C_{N_n} = \frac{\partial}{\partial B} \left[\frac{\partial(C_N \cot \psi)}{\partial J'}\right] B_n J_n \sin \psi_n \qquad (4.9)$$

$$C_{y_n} = \frac{\partial}{\partial B} \left[\frac{\partial(C_y \cot \psi)}{\partial J'}\right] B_n J_n \sin \psi_n + \frac{\partial}{\partial J'} \left[\frac{\partial(C_y \cot)}{\partial J'}\right] J'_n J_n \sin \psi_n \qquad (4.10)$$

$$C_{M_n} = \frac{\partial}{\partial J'} \left[\frac{\partial C_M}{\partial \psi_n}\right] \psi_n J'_n \qquad (4.11)$$

C_{T_n} is the coefficient of thrust (T_n), C_{p_n} is the coefficient of power used to express torque (Q_n), C_{N_n} is the coefficient of normal thrust (N_n)--the thrust component perpendicular to T_n. C_{y_n} and C_{M_n} are the lateral and longitudinal hub moment coefficients that appear during wing tilt.

3. Main Propeller Force and Moment Expressions. Before expressing the force and moment contribution in body axes due to the propellers, the individual propeller forces and moments developed in propeller axes are stated in terms of equations (4.7), (4.8), (4.9), (4.10) and (4.11).

$$T_n = D^4 \left(\frac{N_n}{N_o}\right)^2 \left(\frac{\rho}{\rho_o}\right) C_{T_n} \qquad (4.12)$$

$$N_n = D^4 \left(\frac{N_n}{N_o}\right)^2 \left(\frac{\rho}{\rho_o}\right) C_{N_n} \qquad (4.13)$$

$$Y_n = D^5 \left(\frac{N_n}{N_o}\right)^2 \left(\frac{\rho}{\rho_o}\right) C_{Y_n} \qquad (4.14)$$

$$M_n = D^5 \left(\frac{N_n}{N_o}\right)^2 \left(\frac{\rho}{\rho_o}\right) C_{M_n} \tag{4.15}$$

$$Q_n = \frac{D^5}{2\pi} \left(\frac{N_n}{N_o}\right)^2 \left(\frac{\rho}{\rho_o}\right) C_{P_n} \tag{4.16}$$

In equations (4.12) through (4.16) N_o is the maximum RPM of the propellers and ρ_o is the air density at sea level on a standard day. These equations then enable us to write the propeller force and moment contributions in aircraft body axes. Observe that each equation is subscripted by n so that each propeller individually influences the forces and moments.

$$(\Delta X_a)_p = \sum_{n=1}^{4} (T_n \cos i_w - N_n \cos \beta_n \sin i_w) \tag{4.17}$$

$$(\Delta Y_a)_p = \sum_{n=1}^{4} (-N_n \sin \beta_n) \tag{4.18}$$

$$(\Delta Z_a)_p = \sum_{n=1}^{4} (-T_n \sin i_w - N_n \cos \beta_n \cos i_w) \tag{4.19}$$

$$\begin{aligned}(\Delta L_a) = &- [(\Delta Z_a)_{p_1} - (\Delta Z_a)_{p_4}] y_1 - [(\Delta Z_a)_{p_2} - (\Delta Z_a)_{p_3}] y_2 \\ &- [(\Delta Y_a)_{p_1} + (\Delta Y_a)_{p_4}] z_1 - [(\Delta Y_a)_{p_2} + (\Delta Y_a)_{p_3}] z_2 \\ &- \sum_{n=1}^{4} (Y_n \cos \beta_n) \sin i_w - \sum_{n=1}^{4} (M_n \sin \beta_n) \sin i_w\end{aligned} \tag{4.20}$$

$$\begin{aligned}(\Delta M_a)_p = &M_{T_{PIVOT}} + \sum_{n=1}^{4} T_n(\cos i_w)\, z_{PIVOT} + \sum_{n=1}^{4} T_n(\sin i_w)\, x_{PIVOT} \\ &- (N_1 \cos \beta_1 \sin i_w + N_4 \cos \beta_4 \sin i_w) z_1 \\ &- (N_2 \cos \beta_2 \sin i_w + N_3 \cos \beta_3 \sin i_w) z_2 \\ &+ (N_1 \cos \beta_1 \cos i_w + N_4 \cos \beta_4 \cos i_w) x_1 \\ &+ (N_2 \cos \beta_2 \cos i_w + N_3 \cos \beta_3 \cos i_w) x_2 \\ &- \sum_{n=1}^{4} (Y_n \sin \beta_n) + \sum_{n=1}^{4} (M_n \cos \beta_n)\end{aligned} \tag{4.21}$$

Where $M_{T_{PIVOT}} = 1.625(T_1 + T_4) + 1.092(T_2 + T_3)$

$$(\Delta N_a)_p = -[(\Delta X_a)_{p_1} - (\Delta X_a)_{p_4}]\, y_1 - [(\Delta X_a)_{p_2} - (\Delta X_a)_{p_3}]\, y_2$$

$$+[(\Delta Y_a)_{p_1} + (\Delta Y_a)_{p_4}]\, x_1 + [(\Delta Y_a)_{p_2} + (\Delta Y_a)_{p_3}]\, x_2 \qquad (4.22)$$

$$- \sum_{n=1}^{4} (Y_n \cos \beta_n) \cos i_w - \sum_{n=1}^{4} (M_n \sin \beta_n) \cos i_w$$

In equations (4.17) through (4.22) the letter p denotes the effects of the main propellers and p subscripted p_n where n = 1, 2, 3, 4 is a particular propeller. For example, in $(\Delta L_a)p$ the term $(\Delta Z_a)p_1$ equals $(-T_1 \sin i_w - N_1 \cos \beta_1 \cos i_w)$. In order to better appreciate these equations let us consider the aircraft in normal forward flight where the propeller wind vector is parallel to the x-z plane ($\beta_n = 0$) and there is no tilt of the wing ($i_w = 0$). The equations (4.17) through (4.22) then become:

$$(\Delta X_a)p = \sum_{n=1}^{4} T_n \qquad (4.23)$$

$$(\Delta Y_a)p = 0 \qquad (4.24)$$

$$(\Delta Z_a)p = \sum_{n=1}^{4} - N_n \qquad (4.25)$$

$$(\Delta L_a)p = -(-N_1 + N_4)\, y_1 - (-N_2 + N_3)\, y_2 \qquad (4.26)$$

$$(\Delta M_a)p = M_{T_{PIVOT}} + \sum_{n=1}^{4} T_n\, Z_{PIVOT} \qquad (4.27)$$

$$+ (N_1 + N_4)x_1 + (N_2 + N_3)x_2 + \sum_{n=1}^{4} M_n$$

$$(\Delta N_a)p = -(T_1 - T_4)y_1 - (T_2 - T_3)y_2 - \sum_{n=1}^{4} Y_n \qquad (4.28)$$

Equation (4.23) is the total thrust and (4.25) is the total normal force due to the propellers. Equation (4.26) is the rolling moment contribution which will be zero if outboard (n=1 and 4) and inboard (n=2 and 3) normal propeller forces are balanced; (4.27) is the pitching moment contribution; and (4.28) is the turning moment contribution which will be negligible if the outboard (n=1 and 4) and inboard (n=2 and 3) thrusts are balanced and $\Sigma Y_n = 0$. The incremental propeller forces and moments equations (4.17) through (4.22) will be included in the total aerodynamic forces and moments.

Wing. The calculation of wing aerodynamics forces and moments is complicated by the wing tilt during vertical and transition flight. These forces and moments are developed in wing stability axes by first considering the wing geometry and then defining wing aerodynamic coefficients in accord with Reference 6.

1. Wing Geometry. In wing stability axes there occurs an induced velocity (ΔV) due to the propeller wash across the wing. This gives the effect of increased lift. In order to describe the effect, a coefficient of thrust of the wing ($C_{T,S}$) is defined as a function to total aircraft velocity (V_B).

$$C_{T,S} = \frac{T}{q_w S_p} \tag{4.29}$$

where S_p is the total disk area of the four propellers and q_w is the wing dynamic pressure.

$$q_w = (q + \frac{T}{S_p}), \text{ where } q = 1/2 \ \rho V_B^2$$

At low forward speed during transition and in hover the effect of $C_{T,S}$ is at a maximum. The particular function of $C_{T,S}$ is dependent upon wing tilt and wing flap angle which should be determined from manufacturer's data.

From the geometry in Figure 8 the u_w component of wing velocity is $u_p + \Delta V$. The velocity u_p is the average of the u_n velocities developed in propeller axes and is the predominant velocity effect in wing axes. u_p is then the velocity of the aircraft ($V_B \cos \beta_F$) and as such defines the body axes velocities U and W. Any V in body axes is equal to v_w since we are concerned with wing stability axes and u_p is contained in the x-z body axes and x-z wing axes planes. The total velocity in wing axes (V_w) is as follows:

$$V_w = (w_p^2 + (u_p + \Delta V)^2 + v_w^2)^{1/2}, \text{ where } v_w = V \tag{4.30}$$

From Figure 8, $w_p = w_w = -U \sin i_w + W \cos i_w$ (4.31)

and $u_p = U \cos i_w + W \sin i_w$

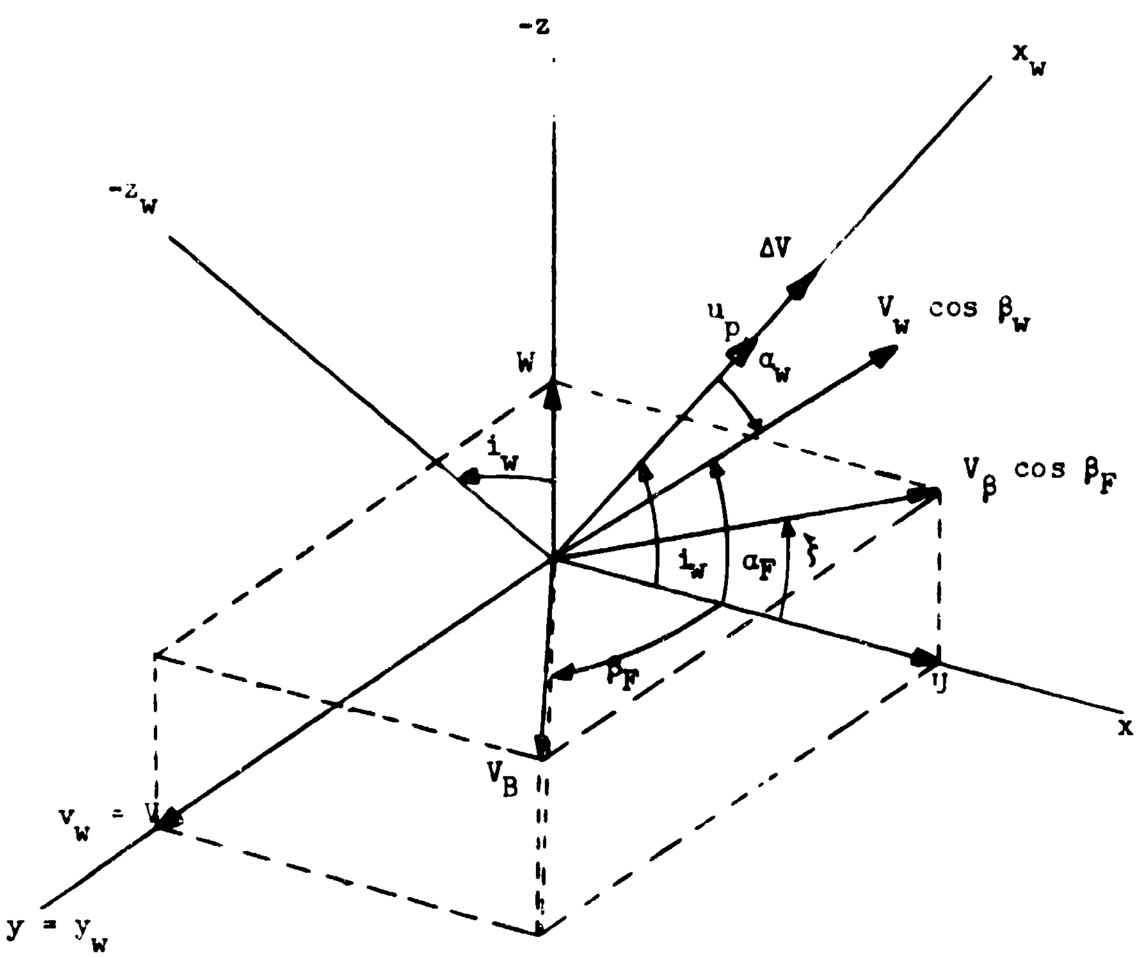


Figure 8. Wing Axes

From the foregoing the wing angle of attack (α_w), and the wing sideslip angle (β_w) can be stated as:

$$\alpha_w = \tan^{-1} \left(\frac{w_p}{u_p + \Delta V}\right), \qquad (4.32)$$

with the sign of α_w as positive down from plus x_w.

$$\beta_w = \tan^{-1} \frac{V}{\sqrt{(u_p + \Delta V)^2 + W_p^2}} \qquad (4.33)$$

The expression for induced velocity (ΔV) is derived from momentum theory.[1] ΔV is similar to the mean inflow velocity ($W_{i_{mean}}$) developed for helicopter inflow analysis.[2]

$$\Delta V = - u_p + \left(u_p^2 + \frac{2T}{\rho S_p}\right)^{1/2} \qquad (4.34)$$

Rearranging we have:

$$(u_p + \Delta V) = \left(u_p^2 + \frac{2T}{\rho S_p}\right)^{1/2}$$

$$\text{but } V_w = u_p + \Delta V \qquad (4.35)$$

and $u_p = V_B$, so that

substituting for ($u_p + \Delta V$) and u_p, and multiplying by $\frac{\rho}{2}$ we have:

$$1/2 \quad \rho V_w^2 = 1/2 \quad \rho V_B^2 + \frac{T}{S_p}$$

$q = 1/2 \quad \rho V^2$, where q is the dynamic pressure

but $q_w = 1/2 \quad \rho V_w^2$

so that we have:

$$q_w = \left(q + \frac{T}{S_p}\right) \qquad (4.36)$$

1. For example, see Airplane Aerodynamics, Reference 5.
2. NAVTRADEVCEN 1205-1, Section 3.

In this manner q_w develops naturally from the momentum theory definition of ΔV. The wing dynamic pressure is the pressure used in equation (4.29) for $C_{T,S}$. Consequently, we have now related the coefficient $C_{T,S}$ with the induced velocity, ΔV, through the wing dynamic pressure, q_w. We are now ready to develop the wing aerodynamic coefficients.

2. Wing Aerodynamic Coefficients. In order to develop the wing forces and moments five aerodynamic coefficients will be defined as in Reference 6. Since the development is in wing axes, rolling (p) and turning (r) rates necessary to define these coefficients are transformed from body axes. The pitching rate (q_1) is the same in wing axes since q_1 lies in the x-z plane and the $x_w - z_w$ plane. The transformation of the angular rates (p and r) is accomplished by a rotation, ξ. From Figure 8 ξ is equal to $(i_w - \alpha_w)$. We can then write for the wing rolling rate (p_w) and the wing turning rate (r_w) the following equations.

$$p_w = p \cos \xi - r \sin \xi \tag{4.37}$$

$$r_w = p \sin \xi + r \cos \xi \tag{4.38}$$

The aerodynamic coefficients for the wing are C_D, C_L, (C_{ℓ_w}), (C_{m_w}) and (C_{n_w}). They are defined as follows in accordance with their development in Reference 6.

$$C_D = C_{D_o} + KC_L^2 + \frac{\partial C_D}{\partial \delta F} \cdot \delta_F + \frac{\partial^2 C_D}{\partial \delta^2 F} \cdot \delta^2 F \tag{4.39}$$

The strong flap dependence ($\delta^2 F$) in the C_D expression above, is due to the importance of flap during transition.

$$C_L = C_{L_o} + C_{L_{\delta F}} \cdot \delta F + C_{L_{\alpha_F}} \cdot \alpha_w + \frac{\partial C_{L_{\alpha_F}}}{\partial \delta F} \cdot \delta F \cdot \alpha_w \tag{4.40}$$

$$(C_{\ell_w}) = C_{\ell_{\beta_F}} \cdot \beta_w + C_{\ell_{\delta A}} \cdot \delta A + \frac{b}{2V_w} C_{\ell_p} \cdot p_w + \frac{b}{2V_w} C_{\ell_r} \cdot r_w \tag{4.41}$$

$$(C_{m_w}) = C_{m_o} + \frac{\partial C_m}{\partial \delta F} \cdot \delta F + \frac{c}{2V_w} C_{m_{q_1}} \cdot q_1 \tag{4.42}$$

$$(C_{n_w}) = C_{n_{\beta_F}} \cdot \beta_w + C_{n_{\delta A}} \cdot \delta A + \frac{b}{2V_w} C_{n_p} \cdot p_w + \frac{b}{2V_w} C_{n_r} \cdot r_w \tag{4.43}$$

3. Wing Force and Moment Expressions. Before writing the force and moment expressions for the wing, equations (4.39) through (4.43) will be transformed to the aircraft body axes through the angle ξ .

The body axes wing coefficients are then:

$$(C_x)_w = - C_D \cos \xi - C_L \sin \xi \tag{4.44}$$

$$(C_z)_w = C_D \sin \xi - C_L \cos \xi \tag{4.45}$$

$$(C_\ell)_w = (C_{\ell_w}) \cos \xi + (C_{n_w}) \sin \xi \tag{4.46}$$

$$(C_m)_w = (C_{m_w}) - \frac{z_{ac}}{c} (C_x)_w + \frac{x_{ac}}{c} (C_z)_w \tag{4.47}$$

$$(C_n)_w = - (C_{\ell_w}) \sin \xi + (C_{n_w}) \cos \xi \tag{4.48}$$

x_{ac} and z_{ac} are the respective distances from the c.g. of the aircraft in body axes to the aerodynamic center (ac) of the wing in the x-z plane. c is the mean aerodynamic chord.

The wing force and moment contributions to the total force and moment equations, (4.44) through (4.48) can be expressed as forces and moments. For example, if we have a coefficient C_x, a force, X_a, is immediately defined as C_x Sq.[3] S is the wing area of the aircraft and q the dynamic pressure. The coefficients in equations (4.44) through (4.48) are immediately expressible in terms of their respective force and moment contributions.

$$(\Delta X_a)_w = (C_x)_w \; S \, [f(C_{T,S}) \; q_w] \tag{4.49}$$

$$(\Delta Z_a)_w = (C_z)_w \; S[f(C_{T,S}) \; q_w] \tag{4.50}$$

$$(\Delta L_a)_w = (C_\ell)_w \; bS \, [f(C_{T,S}) \; q_w] \tag{4.51}$$

$$(\Delta M_a)_w = (C_m)_w \; cS[f(C_{T,S}) \; q_w] \tag{4.52}$$

$$(\Delta N_a)_w = (C_n)_w \; bS[C_{T,S}) \; q_w] \qquad (4.53$$

3. Use of coefficients to express forces and moments can be reviewed in References 2 and 7.

Equations (4.49) through (4.53) are the wing force and moment contributions. $[f(C_{T,S})\ q_w]$ is the dynamic pressure term incorporating the effects of having thrust produced by the propellers and having wash across the wing. Consider what happens if the aircraft is flying and the engines are turned off (T = 0). Then $q_w = q$ from equation (4.36) and $f(C_{T,S}) \approx 1$ from data available in Reference 6. Thus $[f(C_{T,S})\ q_w] \approx q$ with the engines off and equations (4.49) through (4.53) are wing force and moment equations that would be expected to occur in a jet aircraft.

Vertical Stabilizer and Rudder. Forces and moments for vertical tail (vt) and rudder arise from the relative wind pushing against the vertical tail surfaces thereby causing a turning moment due to control input to the rudder. This gives side force, as well as rolling and turning moments.

Wind pushing against the vertical tail and rudder yields a side force $(Y_a)_{vt}$ and rolling moment $(\Delta L_a)_{vt}$ respectively which are non-dimensionalized in terms of aerodynamic coefficients as C_y and C_ℓ. The rolling moment is coupled in rolling velocity p and turning velocity r. We can then write for C_y and C_ℓ the expressions, (4.54) and (4.55).

$$C_y = C_{y_{\beta_F}} \cdot \beta_F + C_{y_{\delta R}} \cdot \delta R \qquad (4.54)$$

C_{y_β} is the change in side force coefficient with changing sideslip angle. It acts as a damping term. $C_{y_{\delta R}}$ is the change in side force coefficient with rudder deflection and represents the controllable term in the expression. This is important in the use of automatic stabilization systems for aircraft.

$$C_\ell = C_{\ell_{\beta_F}} \cdot \beta_F + C_{\ell_{\delta R}} \cdot \delta R + \frac{b}{2V_B} [C_{\ell_p} \cdot p + C_{\ell_r} \cdot r] \qquad (4.55)$$

C_{ℓ_β} is the change in rolling moment with variation in sideslip angle. It is caused by the position of the vertical tail normally above the X axis and other factors as wing sweepback and dihedral. $C_{\ell_{\delta R}}$ is the change in rolling moment coefficient due to rudder deflection. C_{ℓ_p} is the roll damping derivative. C_{ℓ_r} is the change in rolling moment coefficient with change in yawing velocity.

Rudder deflection will give a turning moment $(\Delta N_a)_{vt}$ which can be expressed by the aerodynamic coefficient C_n, and the expression for C_n is:

$$C_n = C_{n_{\beta_F}} \cdot \beta_F + C_{n_{\delta R}} \cdot \delta R + \frac{b}{2V_R}[C_{n_p} \cdot p + C_{n_r} \cdot r] \tag{4.56}$$

C_{n_β} is the weathercock or static directional derivative. $C_{n_{\delta R}}$ is rudder effectiveness, the change in yawing moment coefficient with rudder deflection. C_{n_p} is the change in yawing moment coefficient with varying rolling velocity. C_{n_r} is the yaw damping derivative. The tail is the main contribution to C_{n_r}.

The forces and moments for the vertical tail can then be expressed in the following equations:

$$(\Delta Y_a)_{vt} = C_y S_q(\frac{q_{vt}}{q}) \tag{4.57}$$

$$(\Delta L_a)_{vt} = C_\ell \quad Sq(\frac{q_{vt}}{q}) \tag{4.58}$$

$$(\Delta N_a)_{vt} = C_n bSq(\frac{q_{vt}}{q}) \tag{4.59}$$

Here q is the free stream dynamic pressure and q_{vt} is the vertical tail dynamic pressure. This force and these moments will be included in the total aerodynamic forces and moments.

Horizontal Stabilizer. The equations for forces and moments for the horizontal stabilizer (hs) will be developed in the standard manner.

We define $\alpha_t = \alpha_F + i_t - \epsilon$

Here α_t is the angle of attack of the tail, α_F is the angle of attack of the fuselage, i_t is the angle of incidence of the tail and ϵ is the down wash angle.

In accord with Reference 6 we have

$$\epsilon = [4 - .0421\ (i_w + \alpha_F)]\ [(C_L)_w\ f(C_{T,S})] + i_w - \alpha_w + \alpha_F$$

for the XC-142A.

The lift (C_{L_t}) and (C_{D_t}) coefficients of the tail can then be expressed in the following relations

$$C_{L_t} = C_{L_{\alpha_t}} \cdot \alpha_t$$

$$C_{D_t} = C_{D_{o_t}} + K_t (C_{L_t})^{2} \text{ where } K_t = \frac{1}{\pi A_t}$$ [illegible]

The horizontal stabilizer (hs) can contribute forces in the x and z directions, and a pitching moment. The equations are as follows:

$$(\Delta X_a)_{hs} = - [C_{D_t} \cos (i_t - \alpha_t) + C_{L_t} \sin (i_t - \alpha_t)] \, Sq(\frac{q_{hs}}{q}) \qquad (4.60)$$

$$(\Delta Z_a)_{hs} = - [-C_{D_t} \sin (i_t - \alpha_t) + C_{L_t} \cos (i_t - \alpha_t)] \, Sq(\frac{q_{hs}}{q}) \qquad (4.61)$$

$$(\Delta M_a)_{hs} = - (\Delta X_a)_{hs} \cdot h_{hs} + (\Delta Z_a)_{hs} \cdot l_{hs}$$

$$+ \frac{c^2 S \rho}{4} [C_{m_{q_1}} \cdot q_1 V_B + C_{m_{\dot{\alpha}}} \cdot \dot{W}] \qquad (4.62)$$

l_{hs} is distance from aircraft c.g. to the aerodynamic center (a.c.) of the horizontal stabilizer and h_{hs} is the height of a.c. above the c.g. Both l_{hs} and h_{hs} are measured in the x-z plane of the aircraft body axes. S is the wing area, ρ is the air density, c is the mean aerodynamic chord and the angle $(i_t - \alpha_t)$ is used to transform C_{L_t} and C_{D_t} to body axes.

$C_{m_{q_1}}$ is the pitch damping derivative—the change in pitching moment coefficient as pitch velocity is changed. $C_{m_{\dot{\alpha}}}$ is due to the fact that when the wing undergoes a change in angle of attack, there is a time lag before the change in downwash is felt at the tail surfaces.

$(\Delta X_a)_{hs}$, $(\Delta Z_a)_{hs}$ and $(\Delta M_a)_{hs}$ will be included in the total aerodynamic forces and moments.

Tail Rotor. The z force $(\Delta Z_a)_{TR}$ and the pitching moment $(\Delta M_a)_{TR}$ developed at the tail will be obtained by finding a tail rotor (TR) advance ratio (J_{TR}). From J_{TR} the tail rotor thrust coefficient $(C_{T_{TR}})$ and tail rotor power coefficient $(C_{P_{TR}})$ will be found. In turn, the tail rotor thrust (T_{TR}) and torque (Q_{TR}) is obtained and consequently $(\Delta Z_a)_{TR}$ and $(\Delta M_a)_{TR}$. T_{TR} is positive in the -z direction.

We define the total tail rotor velocity (V_{TR}) as $V_{TR} = [(u_{TR})^2 + (w_{TR})^2]^{1/2}$. Here u_{TR} and w_{TR} are defined as:

$$u_{TR} = V_B \sin (\psi_o)_{TR}$$

$$w_{TR} = V_B \cos (\psi_o)_{TR} - l_{TR} q_1$$

l_{TR} is the distance from the center of the tail rotor hub to the aircraft c.g. and $(\psi_o)_{TR}$ locates the tail rotor with respect to the aircraft body axes.

$$(\psi_o)_{TR} = 90^o + (\alpha_F - \varepsilon)$$

Here ε is again the downwash angle and is defined as for the horizontal stabilizer.

The advance ratio for the tail rotor is

$J_{TR} = \dfrac{60\ V_{TR}}{N_{TR}\ D_{TR}}$ or $J'_{TR} = \dfrac{60(-w_{TR})}{N_{TR}\ D_{TR}}$ where N_{TR} is

the RPM and D_{TR} the diameter of the tail rotor.

We now define $(C_{T_{TR}}')$ and $(C_{p_{TR}})$ as is done in Reference 6.

$$C_{T_{TR}} = C_{T_{TR}} (B_{TR}) + \frac{\partial C_{T_{TR}}}{\partial J'_{TR}} (J'_{TR})$$

$$C_{p_{TR}} = \frac{\partial^2 C_p}{\partial B^2_{TR}} (B_{TR})^2$$

Here B_{TR} is the collective pitch of the tail rotor blades and $C_{T_{TR}}(B_{TR})$ is developed from a curve of $C_{T_{TR}}$ plotted versus B_{TR}.

The thrust (T_{TR}) and torque (Q_{TR}) of the tail rotor is then written as:

$$T_{TR} = D^4_{TR} \left(\frac{\rho}{\rho_o}\right) \left(\frac{N_{TR}}{(N_o)_{TR}}\right)^2 C_{T_{TR}} \tag{4.63}$$

$$Q_{TR} = D^5_{TR} \left(\frac{\rho}{\rho_o}\right)\left(\frac{N_{TR}}{(N_o)_{TR}}\right)^2 C_{p_{TR}} \tag{4.64}$$

Consequently the force and moment terms can be written directly.

$$(\Delta Z_a)_{TR} = T_{TR} \tag{4.65}$$

$$(\Delta M_a)_{TR} = T_{TR} l_{TR} \tag{4.66}$$

$$(\Delta N_a)_{TR} = Q_{TR} \tag{4.67}$$

Fuselage. In a very direct manner we can write the effects of the fuselage (F) on the total aerodynamic forces and moments.

We have for the forces

$$(\Delta X_a)_F = -\frac{1}{2}\rho V_B^2 S C_{D_o} \tag{4.68}$$

C_{D_o} is the equilibrium drag coefficient.

$$(\Delta Y_z)_F = +\frac{1}{2}\rho V_B^2 S C_{y_{\beta_F}} \cdot \beta_F \tag{4.69}$$

$C_{y_{\beta_F}}$ is the change in side force with respect to a changing sideslip angle.

$$(\Delta Z_a)_F = -\frac{1}{2}\rho V_B^2 S C_{L_{\alpha_F}} \cdot \alpha_F \tag{4.70}$$

$C_{L_{\alpha_F}}$ is the change in lift coefficient with varying angle of attack. This is also known as the lift curve slope.

We have for the moments:

$$(\Delta L_a)_F = 0$$

$$(\Delta M_a)_F = \frac{1}{2}\rho V_B^2 S c C_{m_o} + \frac{1}{2}\rho V_B^2 Sc C_{m_{\alpha_F}} \cdot \alpha_F$$

C_{m_o} is the aerodynamic pitching moment coefficient in equilibrium flight and $C_{m_{o_F}}$ is the longitudinal static stability derivative.

$$(\Delta N_a)_F = \frac{1}{2}\rho V_B^2 S b C_{n_{\beta_F}} \cdot \beta_F \tag{4.71}$$

$C_{n_{\beta_F}}$ is the static directional or "weathercock" derivative.

In the above expressions $q = \frac{1}{2} \rho V_B^2$ which is the dynamic pressure. b is the wing span, c is the mean aerodynamic chord and S is the wing area.

For hovering and in transition regions β_F is assumed small so that $\cos \beta_F = 1$.

EQUATIONS OF MOTION - XC-142A. The aerodynamic force and moment terms developed for each of the major aircraft components will be combined and finally expressed in the equations of motion for the total aircraft.

Total Aerodynamic Forces and Moments. The total aerodynamic forces and moments are as follows:

$$X_a = (\Delta X_a)_p + (\Delta X_a)_w + (\Delta X_a)_{hs} + (\Delta X_a)_F$$

$$Y_a = (\Delta Y_a)_p + (Y_a)_{vt} + (\Delta Y_a)_F$$

$$Z_a = (\Delta Z_a)_p + (\Delta Z_a)_w + (\Delta Z_a)_{hs} + (\Delta Z_a)_{TR} + (\Delta Z_a)_F$$

$$L_a = (\Delta L_a)_p + (\Delta L_a)_w + (\Delta L_a)_{vt}$$

$$M_a = (\Delta M_a)_p + (\Delta M_a)_w + (\Delta M_a)_{hs} + (\Delta M_a)_{TR} + (\Delta M_a)_F$$

$$N_a = (\Delta N_a)_p + (\Delta N_a)_w + (\Delta N_a)_{vt} + (\Delta N_a)_{TR} + (\Delta N_a)_F$$

Equations of Motion Expanded. The forces and moments are presented in body axes. These equations are subject to simplification in representation of aerodynamic propeller forces and moments, when flight test data are available from which to determine typical magnitudes of terms.

1. X Force Equation

$$(C_{x_w} \mathcal{E} f(C_{T,S}) q_w + \sum_{n=1}^{4} (T_n \cos i_w - N_n \cos \beta_n \sin i_w)$$

$$-[C_{D_t} \cos (i_t - \alpha_t) + C_{L_t} \sin (i_t - \alpha_t)] \, Sq(\frac{q_{hs}}{q}) - Sq \, C_{D_o}$$

$$= m(\dot{U} + Wq_1 - Vr) + mg \sin \theta$$

2. Y Force Equation.

$$\sum_{n=1}^{4} (-N_n \sin \beta_n) + C_\ell \, bSq \left(\frac{q_{vt}}{q}\right) + Sq \, C_{y_{\beta_F}} \cdot \beta_F$$

$$= m(\dot{V} + Ur - Wp) - mg \cos \theta \sin \Phi$$

3. Z Force Equation.

$$(C_z)_w Sf(C_{T,S}) q_w + \sum_{n=1}^{4} (-T_n \sin i_w - N_n \cos \beta_n \cos i_w)$$

$$+ [C_{D_t} \sin(i_t - \alpha_t) + C_{L_t} \cos (i_t - \alpha_t)] \, Sq\left(\frac{q_{hs}}{q}\right) + T_{TR} - SqC_{L_{\alpha_F}} \cdot \alpha_F$$

$$= m(\dot{W} + Vp - Uq_1) - mg \cos \theta \cos \Phi$$

4. Roll Equation.

$$(C_l)_w bSf(C_{T,S}) q_w$$

$$+ [(+T_1 \sin i_w + N_1 \cos \beta_1 \cos i_w) - (+T_4 \sin i_w + N_4 \cos \beta_4$$

$$\cos i_w)] \, y_1$$

$$+ [(+T_2 \sin i_w + N_2 \cos \beta_2 \cos i_w) - (+T_3 \sin i_w + N_3 \cos \beta_3$$

$$\cos i_w)] \, y_2$$

$$+ (+N_1 \sin \beta_1 + N_4 \sin \beta_4) z_1 + (+N_2 \sin \beta_2 + N_3 \sin \beta_3) \, z_2$$

$$- \sum_{n=1}^{4} (Y_n \cos \beta_n) \sin i_w - \sum_{n=1}^{4} (M_n \sin \beta_n) \sin i_w + C_\ell \, b \, Sq\left(\frac{q_{vt}}{q}\right)$$

$$= I_{11}\dot{p} + I_{13}(\dot{r} + pq_1) + (I_{33} - I_{22}) \, q_1 r + (I_E \dot{\Omega}_E) \cos i_w$$

$$- q_1 (I_E \Omega_E) \sin i_w + q_1 \, I_{TR} \Omega_{TR}$$

5. Pitch Equation.

$$(C_m)_w cSf(C_{T,S}) q_w$$

$$+ M_{T_{PIVOT}} + \sum_{1}^{4} T_n (\cos i_w) \, z_{pivot} + \sum_{1}^{4} T_n \, (\sin i_w) \, x_{pivot}$$

$$- (N_1 \cos \beta_1 \sin i_w + N_4 \cos \beta_4 \sin i_w)\, z_1$$

$$- (M_2 \cos \beta_2 \sin i_w + N_3 \cos \beta_3 \sin i_w)\, z_2$$

$$+ (N_1 \cos \beta_1 \cos i_w + N_4 \cos \beta_4 \cos i_w)\, x_1$$

$$+ (N_2 \cos \beta_2 \cos i_w + N_3 \cos \beta_3 \cos i_w)\, x_2 - \sum_{n=1}^{4} (Y_n \sin \beta_n)$$

$$+ \sum_{n=1}^{4} (M_n \cos \beta_n)$$

$$+ [C_{D_t} \cos (i_t - \alpha_t) + C_{L_t} \sin (i_t - \alpha_t)]\, Sq(\frac{q_{hs}}{q}) \cdot h_{hs}$$

$$+ [C_{D_t} \sin (i_t - \alpha_t) - C_{L_t} \cos (i_t - \alpha_t(]\, Sq(\frac{q_{hs}}{q}) \cdot l_{hs}$$

$$+ \frac{c^2 S\rho}{4} [C_{m_{q_1}} \cdot q_1 V_B + C_{m_{\dot{\alpha}}} \cdot \dot{W}] + T_{TR} \cdot l_{TR} + qSc(C_{m_o} + C_{m_{\alpha_F}} \cdot \alpha_F)$$

$$= I_{22}\dot{q} + I_{13}(r^2 - p^2) + (I_{11} - I_{33})pr$$

$$+ p(I_E \Omega_E) \sin i_w + r(I_E \Omega_E) \cos i_w$$

6. **Yaw Equation.**

$$(C_n)_w bSf(C_{T,S})q_w$$

$$- [(T_1 \cos i_w - N_1 \cos B_1 \sin i_w) - (T_4 \cos i_w - N_4 \cos \beta_4 \sin i_w]y_1$$

$$- [(T_2 \cos i_w - N_2 \cos \beta_2 \sin i_w) - (T_3 \cos i_w - N_3 \cos \beta_3 \sin i_w]y_2$$

$$- (+N_1 \sin \beta_1 + N_4 \sin \beta_4)x_1 - (+N_2 \sin \beta_2 + N_3 \sin \beta_3)x_2$$

$$- \sum_{n=1}^{4} (Y_n \cos \beta_n) \cos i_w - \sum_{n=1}^{4} (M_n \sin \beta_n) \cos i_w$$

$$+ C_n\, bSq(\frac{q_{vt}}{q}) + q\, Sb\, C_{n_{\beta_F}} \cdot \beta_F + Q_{TR}$$

$$= I_{33}\dot{r} + I_{13}(\dot{p} - q_1 r) + (I_{22} - I_{11})pq_1$$

$$- (I_E \dot{\Omega}_E) \sin i_w - q_1(I_E \Omega_E) \cos i_w - pI_{TR}\Omega_{TR}$$

No simplifications have been made on the right side of these six equations. External stores, rough air, or landing gear conditions have not been developed in these equations.

TILT-DUCT

The VZ-4DA VTOL aircraft of DOAK Aircraft will be used in the example of simulation equations of motion for a tilt-duct aircraft. Much of the development which follows has been based upon information gleaned from References 1 and 8. Soule and Baumgarten in Reference 8 present a method for calculating the aerodynamic forces and moments for the VZ-4DA and Breul in Reference 1 presents a study of handling qualities for a tilt-duct aircraft. From this information and the presentation set forth for the tilt-wing example, a well-defined set of engineering simulation equations for the tilt-duct will be presented realizing that an actual simulation will be dependent upon the type and usefulness of the aerodynamic data available from the manufacturer of the aircraft.

In order that we may have some idea of the specific details of the VA-4DA, Figure 2 is a three-view drawing of the VZ-4DA, Table 4 is a selected list of physical characteristics of the aircraft and Table 5 is a list of symbols used in developing the equations of motion. As much as possible, repetition of symbols will be used in the various examples of V/STOL aircraft in this section of the report.

From this general idea of the VZ-4DA, we may now develop the mathematical model for the VZ-4DA. First, define, if necessary, applicable axis systems. Second, develop the aerodynamic coefficients. Finally, write the equations of motion for the VZ-4DA. As needed we will borrow from the results obtained in the tilt-wing equations of motion and from the results obtained in References 1 and 8.

AXIS SYSTEMS FOR THE VZ-4DA. In order to describe the dynamics and aerodynamics of the tilt-duct aircraft, one axis system, the duct-axis system, in addition to the inertial, body, stability and wind axes is required. The wind axis system is complicated by the tilt of the ducts. Both of the ducts tilt through the same angle simultaneously. Figure 9 shows the wing tilt angle (i_W) and the duct incidence angle (i_D). Duct incidence is described in the aircraft stability axes. The force and moment coefficients for the ducted fans are described in the duct axis system. There are two duct-axis systems--one located at each duct. This is similar to the repeated propeller axis systems developed for the XC-142A. We define the duct axis system by the following three mutually orthogonal velocity vectors. First, there is a side duct velocity (V_S) parallel to the y body axis and in a minus y direction. Second, there is an axial velocity (V_A) into the center of the duct. Finally, there is a normal velocity (V_R) parallel to the x-z plane of the aircraft body axis system and perpendicular to V_S and V_A. If there is no duct incidence, the duct-axis systems as shown in Figure 10 are similar to the x, y, z body axes. The duct-axes (x_{D1}, y_{D1}, z_{D1} and x_{D2}, y_{D2}, z_{D2}) are right-handed systems with y parallel to y_{D1} and y_{D2} and x in the same direction as x_{D1} and x_{D2} and z in the same direction as z_{D1} and z_{D2} when i_W is zero.

Airplane:

Over-all length	31.2 ft.
Over-all span (including ducts)	25.5 ft.
Over-all height	10.3 ft.

Propulsive Device:

Fan

Type	fixed pitch
Number of blades	8
Diameter	4 ft.
Blade root chord	0.312 ft.
Blade tip chord	0.184 ft.
Solidity	0.117
Blade angle (.75R)	0.306 radians
Hub to tip diameter ratio	0.333
Fan location, position of hub from duct lip	0.806 ft.

Duct

Centerbody

Diameter (maximum)	1.333 ft.
Over-all length	5.79 ft.

Trim Vane

Area	5.66 $ft.^2$
Span	4.525 ft.
Airfoil section	NACA 0009
Flap chord, 25 percent	0.313 ft
Movement	0 to + 0.349 radians

Inlet Guide Vanes

Number of vanes	14
Area (per vane)	0.333 ft.
Span	1.333 ft.
Chord (constant)	0.25 ft.
Airfoil section	NACA 65^2 - 015
Vane location, position of C/4 from duct lip	0.4 ft.
Movement	$\pm$ .1745 radians

Wing Group:

Wing

Area (excluding ducts)	96.0 $ft.^2$
Span (excluding ducts)	16.0 ft.
Mean aerodynamic chord	6.08 ft.
Aspect ratio	2.67
Airfoil section	NACA 2418 (mod.)
Dihedral	0 radians

Table 4. Selected Aircraft Characteristics - VZ-4DA

Aileron	
Area (per aileron)	6.0 $ft.^2$
Span	3.0 ft.
Movement	± .262 radians
Empenage Group:	
Horizontal Tail	
Area (projected)	28.5 $ft.^2$
Span (projected)	11.8 ft.
Mean aerodynamic chord	2.5 ft.
Aspect ratio	4.9
Airfoil section	NACA 0012 (Mod.)
Incidence	variable, 0 to + 0.1745
Dihedral	0.1745 radians
Elevator movement	
(stabilizer incidence, 0 radians)	+0.472 radians (T.E. down)
	-0.472 radians (T.E. up)
(stabilizer incidence, +1.745 radians)	+0.297 radians (T.E. down)
	-0.576 radians (T.E. up)
Vertical Tail	
Area	13.9 $ft.^2$
Span	5.2 ft.
Mean aerodynamic chord	2.8 ft.
Aspect ratio	1.95 (geometric); 3.02 (effective)
Airfoil section	NACA 0012 (Mod.)
Fuselage:	
Length (excluding reaction control vanes)	29.3 ft.
Maximum height (approx.)	4.12 ft.
Maximum width	3.0 ft.
Reaction Control Vanes	
Area (within 15 inch tail pipe diameter)	1.042 $ft.^2$
Span (within 15 inch tail pipe diameter)	1.25 ft.
Aspect ratio (effective)	1.32
Airfoil section	NACA 0009
Movement	± 0.612 radians

Table 4. Selected Aircraft Characteristics - VZ-4DA (Cont'd.)

SYMBOL	DEFINITION
A	Aspect ratio
c_T	Thrust coefficient (for velocity = v_D)
$\hat{C}_T$	Thrust coefficient (for velocity = V_D)
$\overline{C}_N$	Normal force coefficient = normal force/$\rho n^2 D^4$
$\overline{C}_S$	Side force coefficient = side force/$\rho n^2 D^4$
$\overline{C}_M$	Pitching moment coefficient = Pitching Moment/$\rho n^2 D^5$
$\overline{C}_\ell$	Total rolling moment coefficient for ducted-fan
$\overline{C}_m$	Total pitching moment coefficient for ducted-fan
$\overline{C}_n$	Total yawing moment coefficient for ducted-fan
C_L	Lift Coefficient
C_D	Drag Coefficient
c_v	Duct chord
c	Wing chord
D	Fan diameter
g	Gravitational acceleration
i_D	Ducted fan incidence angle
i_W	Wing incidence angle
i_{HT}	Horizontal tail incidence angle
i_{VT}	Vertical tail incidence angle
$\hat{J}$	Axial advance ratio (V_D/D)
$\dot{m}$	Mass flow rate (slugs/sec.)
N	Fan speed (R.P.S.)
p	Roll rate about X-Axis (Rad./Sec.)
p_w	Roll rate about v_w (Rad./Sec.)
b_v	Duct span

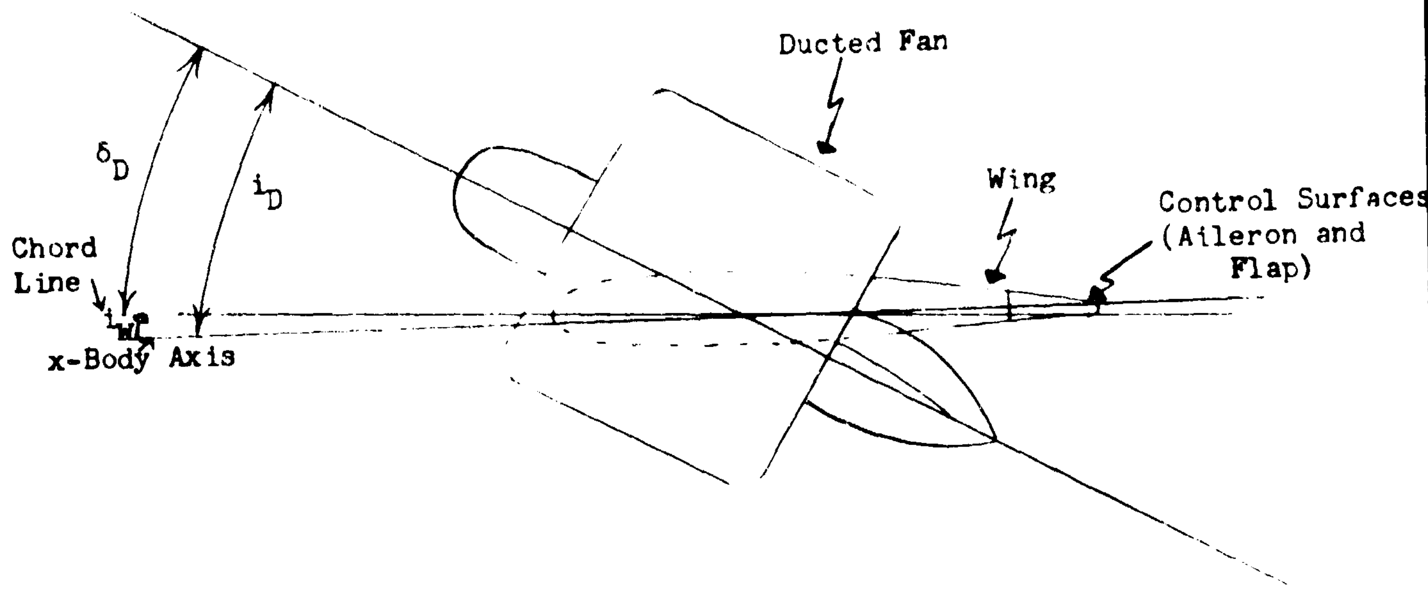
Table [illegible]. Definition of Symbols - VZ-4DA

SYMBOL	DEFINITION
b	Wing span
q_1	Pitch rate about Y-Axis (Rad./Sec.)
r	Yaw rate about Z-Axis (Rad./Sec.)
S	Area - Wing
T_R	Residual Thrust
V_{A-}	Axial velocity into duct
V_{R-}	Side velocity at duct entrance parallel to X-Z plane
V_{S-}	Side velocity at duct entrance parallel to Y-Axis
V_E	Jet exhaust velocity (see Appendix G)
$\bar{x}$	Moment arm from center of gravity along X-Axis
$\bar{y}$	Moment arm from center of gravity along Y-Axis
$\bar{z}$	Moment arm from center of gravity along Z-Axis
α	Angle of attack
α_D	Arc tan (V_{R-}/V_{A-})
β	Sideslip angle
β_D	Arc tan (V_{S-}/V_{A-})
δ_D	Angular displacement
$\pm\epsilon$	Downwash (+) or upwash (-)
ρ	Mass density of air
ρ_E	Mass density of jet efflux
σ	Sidewash angle
D	Duct (Subscript)

Table 5. Definition of Symbols (Cont'd.)

SYMBOL	DEFINITION
E	Elevator
F	Fan, fuel or fuselage
HT	Horizontal tail
I	Interference
IV	Inlet vanes
j	Denotes port (j = 1) or starboard (j = 2)
M.G.C.	Mean geometric chord
N	Centerbody nose, nose or normal
OL	Zero lift
R	Radial or rudder
RC	Reaction control
T	Test or thrust
TV	Trim vane
VH	Horizontal vane
VT	Vertical tail
VV	Vertical vane
W	Wing

Table 5. Definition of Symbols (Cont'd.)



$i_D = i_W + \delta_D$

i_W ≡ wing incidence angle

δ_D ≡ change in duct angle

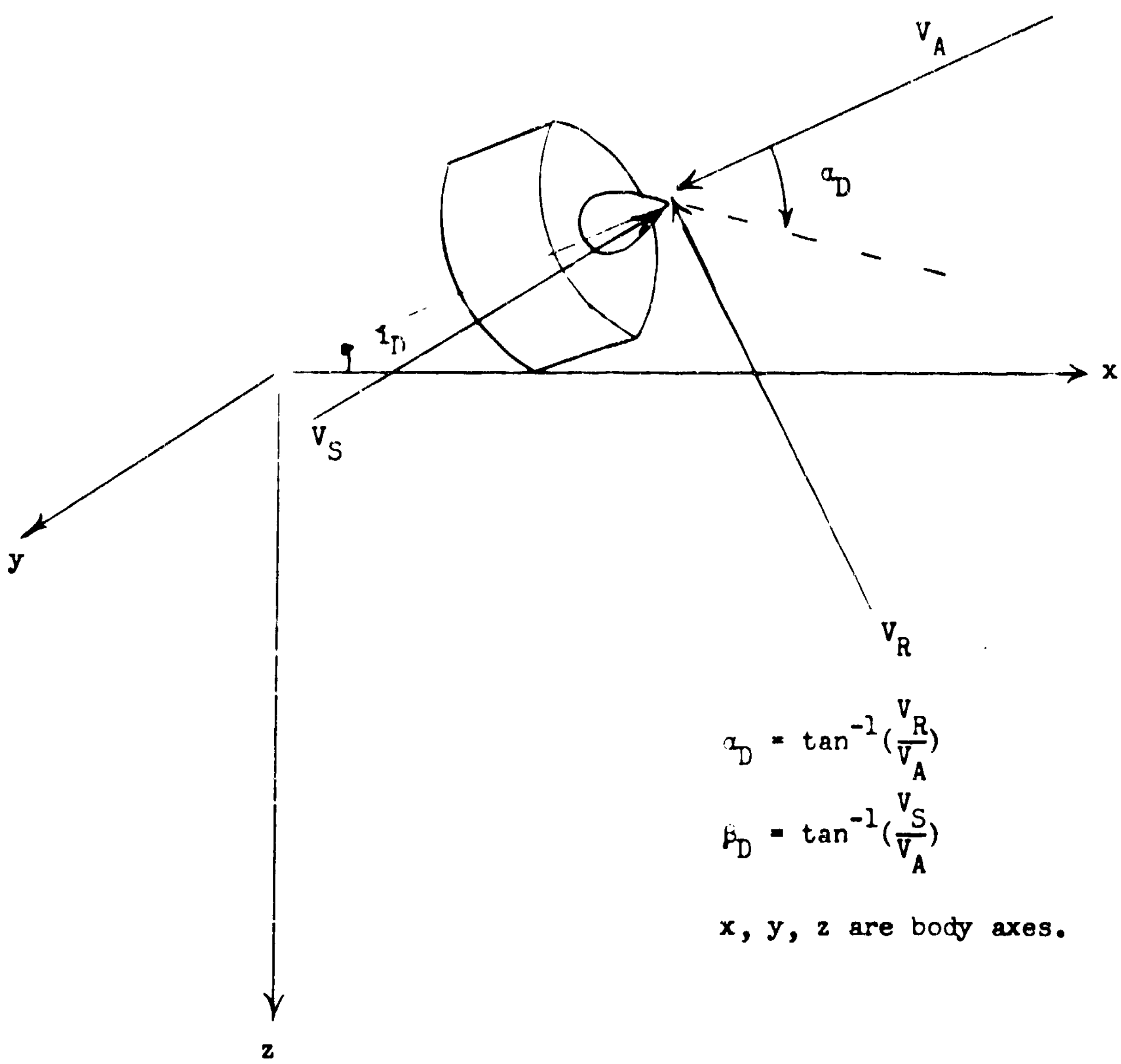
i_D ≡ duct incidence angle

Figure 9. Duct Tilt Axis

V_A Axial velocity into duct

V_R parallel to x-z plane

V_S parallel to y-axis



$$\alpha_D = \tan^{-1}\left(\frac{V_R}{V_A}\right)$$

$$\beta_D = \tan^{-1}\left(\frac{V_S}{V_A}\right)$$

x, y, z are body axes.

Figure 10. Duct Fan Axis

Before developing the aerodynamic forces and moments it should be noted that the VA-4DA has gyroscopic effects due to the rotating masses in the duct fan. In the following expressions i_w is the fixed wing incidence angle and δ_D is the variable duct angle. For the ducted fans we have:

$$+ (I_E\dot{\Omega}_E) \cos(i_w + \delta_D) - q_1(I_E\Omega_E) \sin(i_w + \delta_D) \qquad \text{for } L_a \text{ term.}$$

$$+ p(I_E\Omega_E) \sin(i_w + \delta_D) + r(I_E\Omega_E) \cos(i_w + \delta_D) \qquad \text{for } M_a \text{ term.}$$

$$- (I_E\dot{\Omega}_E) \sin(i_w + \delta_D) - q_1(I_E\Omega_E) \cos(i_w + \delta_D) \qquad \text{for } N_a \text{ term.}$$

These expressions will be added to the internal moments in the final equations for the VZ-4DA. Next the forces and moments-- X_a, Y_a, Z_a, L_a, M_a, and N_a will be developed for the tilt-duct so that a complete set of equations of motion can be written.

AERODYNAMIC FORCES AND MOMENTS VZ-4DA. In order to develop expressions for the aerodynamic external forces and moments, contributions from the major airframe components will be considered separately. The major components to be considered are the wing, the ducted fans, the vertical stabilizer and rudder, the horizontal stabilizer and elevator, the fuselage and the reaction controls. As with the XC-142A after the aerodynamic force and moment expressions are developed for each of these major components, they will be summed to get the total aerodynamic force and moment expressions.

Wing. The calculation of wing aerodynamic forces and moments is relatively straightforward. In Reference 8 a definition of the wing aerodynamic coefficients is developed. These coefficients contain interference and damping effects. The aerodynamic coefficients for the wing in accordance with Reference 8 are as follows:

$$(C_D)_w = (C_{D_o})_w + \frac{\partial C_D}{\partial C_L^2} \, C_{L_\alpha}^2 \cdot \alpha_w \tag{4.72}$$

$$(C_L)_w = (C_{L_o})_w + (C_{L_\alpha}) \cdot \alpha_w \tag{4.73}$$

$$(C_y)_w = (C_{y_\beta})_w \cdot \beta_w + (C_{y_p})_w \frac{b}{2V_w} \cdot p_w + (C_{y_r})_w \frac{b}{2V_w} \cdot r_w \tag{4.74}$$

$$(C_\ell)_w = (C_{\ell_\beta}) \cdot \beta_w + \frac{b}{2V_w} (C_{\ell_p})_w \cdot p_w + \frac{b}{2V_w} (C_{\ell_r})_w \cdot r_w + C_{\ell_{\delta A}} \cdot \delta A \tag{4.75}$$

$$(C_m)_w = (C_{m_o})_w + \frac{c}{2V_w} (C_{m_{q_1}})_w \cdot q_1 + \frac{\partial C_m}{\partial \delta F} \cdot \delta F \tag{4.76}$$

$$(C_n)_w = (C_{n_\beta})_w \cdot \beta_w + \frac{b}{2V_w}(C_{n_p})_w \cdot p_w + \frac{b}{2V_w}(C_{n_r})_w \cdot r_w + C_{n_{\delta A}} \cdot \delta A \quad (4.77)$$

In Figure 8 we see how α_w is defined. For purposes of clarity the wing incidence angle, i_w, has been exaggerated. ΔV is the induced velocity previously defined in equation (4.34) which is derived from wash effects across the wing. In Figure 8, we see that:

$$V_w^2 = (u_w + \Delta V)^2 + v_w^2 + w_w^2 .$$

So that:

$$\alpha_w = \tan^{-1}\left(\frac{w_w}{u_w + \Delta v_w}\right) = \sin^{-1} \frac{w_w}{V_w \cos \beta_w} \quad (4.78)$$

and:

$$\beta_w = \tan^{-1}\left[\frac{v_w}{\sqrt{(u_w + \Delta v)^2 + w_w^2}}\right] \quad (4.79)$$

In equation (4.78) and (4.79) $v_w = V$ and at low velocities where $\Delta V \not\approx 0$, then $V_w = V_B$.

In order to transform wing data to body axis we rotate through an angle where $\xi = (i_w - \alpha_w)$.

Before writing force and moment expressions for the wing, equations (4.72) through (4.77) are transformed to body axes by rotating through the angle ξ as in the similar analysis for the XC-142A (Section IV).

$$(C_x)_w = (C_L)_w \sin \xi - (C_D)_w \cos \xi \quad (4.80)$$

$$(C_z)_n = -(C_L)_w \cos \xi - (C_D)_w \sin \xi \quad (4.81)$$

$$(C_\ell)_w = (C_\ell)_w \cos \xi - (C_n)_w \sin \xi - (C_y)_w \left(\frac{z_{a.c.}}{c}\right) \quad (4.82)$$

$$(C_m)_w = (C_m)_w + \frac{z_{a.c.}}{c}(C_x)_w - \frac{x_{a.c.}}{c}(C_z)_w \quad (4.83)$$

$$(C_n)_w = (C_n)_w \cos \xi + (c_\ell)_w \sin \xi + (C_y)_w \left(\frac{x_{a.c.}}{c}\right) \quad (4.84)$$

$x_{a.c.}$ and $z_{a.c.}$ are the respective distances from the c.g. of the aircraft in aircraft body axes to the aerodynamic center (a.c.) of the wing in the x-z plane. c is the mean aerodynamic chord.

As in the tilt-wing example, (Section IV) Equations (4.80) through (4.84) are readily expressed as force and moment contributions of the wing.

$$(\Delta X_a)_w = (C_x)_w \, S \, q_w \tag{4.85}$$

$$(\Delta Y_a)_w = (C_y)_w \, S \, q_w \tag{4.86}$$

$$(\Delta Z_a)_w = (C_z)_w \, S \, q_w \tag{4.87}$$

$$(\Delta L_a)_w = (C\ )_w \, b \, S \, q_w \tag{4.88}$$

$$(\Delta M_a)_w = (C_m)_w c \, S \, q_w \tag{4.89}$$

$$(\Delta N_a)_w = (C_n)_w \, b \, S \, q_w \tag{4.90}$$

In equations (4.85) through (4.90) S is the wing area, c is the mean aerodynamic chord, b is the wing span and q_w is the wing dynamic pressure ($q_w = 1/2 \, \rho \, V^2_w$). These equations will be incorporated in the total force and moment equations.

Ducted Fan. In the calculation of ducted fan aerodynamic forces and moments wake, interference and damping are included in the definition of the aerodynamic coefficients. Six coefficients $\bar{C}_{T_j}$, $\bar{C}_{N_j}$, $\bar{C}_{S_j}$, $\bar{C}_{\mathcal{M}_j}$, $\bar{C}_{\mathcal{L}_j}$, and $\bar{C}_{\mathcal{N}_j}$ are defined in Reference 8 to describe the behavior of the ducted fan. These coefficients are as follows:

$$\bar{C}_{T_j} = \bar{C}_{T_{T_j}} + \bar{C}_{T_{I_j}} + \Delta\bar{C}_{T_{IV_j}} \tag{4.91}$$

$\bar{C}_{T_j}$ is the total thrust coefficient. $\bar{C}_{T_{T_j}}$ is the change in thrust due to varying thrust, $\bar{C}_{T_{I_j}}$ is the change in thrust due to varying interference effects and $\Delta\bar{C}_{T_{IV_j}}$ is the change in thrust due to the inlet vanes to the duct. $j=1$ for the port duct and $j=2$ for the starboard duct.

$$\bar{C}_{N_j} = \bar{C}_{N_{F_j}} + \bar{C}_{N_{D_j}} + \bar{C}_{N_{I_j}} + \bar{C}_{N_{N_j}} + \bar{C}_{N_{IV_j}} + \bar{C}_{N_{TV_j}} \tag{4.92}$$

$\bar{C}_{N_j}$ is the total normal force coefficient. $\bar{C}_{N_{F_j}}$ results from fan effects, $\bar{C}_{N_{D_j}}$ results from duct effects, $\bar{C}_{N_{I_j}}$ comes from interference effects, $\bar{C}_{N_{N_j}}$ is the change in normal force due to varying normal force, $\bar{C}_{N_{IV_j}}$ is the change in normal force due to the inlet vanes, and finally $\bar{C}_{N_{TV_j}}$ is the change in normal force due to the trim vanes.

$$\bar{C}_{S_j} = \bar{C}_{S_{F_j}} + \bar{C}_{S_{D_j}} + \bar{C}_{S_{N_j}} + \bar{C}_{S_{IV_j}} \tag{4.93}$$

$\bar{C}_{S_j}$ is the side force coefficient. $\bar{C}_{S_{F_j}}$ results from fan effects, $\bar{C}_{S_{D_j}}$ results from duct effects, $\bar{C}_{S_{N_j}}$ is the change in side force due to varying normal force and $\bar{C}_{S_{IV_j}}$ is the change in side force due to inlet vanes.

$$\bar{C}_{m_j} = \bar{C}_{M_{F_j}} + \bar{C}_{M_{D_j}} + \bar{C}_{M_{I_j}} + \bar{C}_{M_{N_j}} + \bar{C}_{M_{IV_j}} + \bar{C}_{M_{TV_j}} \tag{4.94}$$

$\bar{C}_{m_j}$ is the total pitching moment coefficient for the ducted fan.

$\bar{C}_{M_{F_j}}$, $\bar{C}_{M_{D_j}}$, $\bar{C}_{M_{I_j}}$, result respectively from fan, duct and interference effects.

$\bar{C}_{M_{N_j}}$ is the change in pitching moment due to varying normal force.

$\bar{C}_{M_{IV_j}}$ and $\bar{C}_{M_{TV_j}}$ are respectively the changes in pitching moment due to inlet and trim vanes.

$$\bar{C}_{\mathcal{L}_j} = -(-1)^j \left(\frac{\bar{y}_D}{D}\right) [\bar{C}_{T_j} \sin i_D + \bar{C}_{N_j} \cos i_D] - \left(\frac{1}{2} \sin\beta_{D_j}\right)(\hat{C}_{T_{D_j}} \sin i_D)$$

$$- \left[\left(\frac{\bar{x}_F}{D}\right) \bar{C}_{S_{F_j}} + \left(\frac{\bar{x}_D}{D}\right) \bar{C}_{S_{D_j}} + \left(\frac{\bar{x}_{IV}}{D}\right) C_{S_{IV_j}}\right] \sin i_D + K_1 \sin i_d \tag{4.95}$$

$\bar{C}_{\mathcal{L}_j}$ is the total rolling moment coefficient for the ducted fan.

K_1 is a function of axial advance ratio of the ducted fan.

$$\bar{C}_{\eta_j} = -(-1)^j\left(\frac{\bar{y}_D}{D}\right)[\bar{C}_{T_j}\cos i_D - \bar{C}_{N_j}\sin i_D] - \left(\frac{1}{2}\sin \beta_{D_j}\right)(\hat{C}_{T_{D_j}}\cos i_D)$$

$$- \left[\left(\frac{\bar{x}_F}{D}\right)\bar{C}_{S_{F_j}} + \left(\frac{\bar{x}_D}{D}\right)\bar{C}_{S_{D_j}} + \left(\frac{\bar{x}_{IV}}{D}\right)C_{S_{IV_j}}\right]\cos i_D + K_1 \cos i_D \qquad (4.96)$$

$\bar{C}_{\eta_j}$ is the total yawing moment coefficient for the ducted fan.

Before writing the force and moment equations these coefficients are re-expressed as coefficients representing both ducted fans.

$$(\bar{C}_x)_D = K_2 \sum_{j=1}^{2} (\bar{C}_{T_j}\cos i_D - \bar{C}_{N_j}\sin i_D) \qquad (4.97)$$

$$(\bar{C}_z)_D = - K_2 \sum_{j=1}^{2} (\bar{C}_{T_j}\sin i_D + \bar{C}_{N_j}\cos i_D) \qquad (4.98)$$

$$K_2 = \left(\frac{2D^2}{\pi^2 S}\right)$$

$$(C_y)_D = K_2 \sum_{j=1}^{2} \bar{C}_{S_j} \qquad (4.99)$$

$$(\bar{C}_\ell)_D = - K_2\, D/b \sum_{j=1}^{2} \bar{C}_{\mathcal{L}_j} \qquad (4.100)$$

$$(\bar{C}_m)_D = K_2\, D/c \sum_{j=1}^{2} \bar{C}_{m_j} \qquad (4.101)$$

$$(\bar{C}_n)_D = K_2\, D/b \sum_{j=1}^{2} \bar{C}_{\eta_j} \qquad (4.102)$$

The equations for duct forces and moments are then as follows:

$$(\Delta X_a)_D = K_3(\bar{C}_x)_D \qquad (4.103)$$

$$(\Delta Y_a)_D = K_3(\bar{C}_y)_D \qquad (4.104)$$

$$(\Delta Z_a)_D = K_3 (\bar{C}_z)_D \quad K_3 = \frac{\rho}{2} (\pi_n D^2) S \tag{4.105}$$

$$(\Delta L_a)_D = K_3 b (\bar{C}_\ell)_D \tag{4.106}$$

$$(\Delta M_a)_D = K_3 c (\bar{C}_m)_D \tag{4.107}$$

$$(\Delta N_a)_D = K_3 b (\bar{C}_n)_D \tag{4.108}$$

Equations (4.103) to (4.108) will be included in the total aerodynamic force and moment expressions.

Vertical Stabilizer and Rudder. Forces and moments for the vertical tail and rudder will be defined in the same manner as for the XC-142A and Reference 8. The associated coefficients are as follows:

$$C_y = C_{y_\beta} \cdot \beta + C_{y_{\delta R}} \cdot \delta R \tag{4.109}$$

$$C_\ell = C_{\ell_\beta} \cdot \beta + C_{\ell_{\delta R}} \cdot \delta R + \frac{pb}{2V_B} C_{\ell_p} + \frac{rb}{2V_B} C_{\ell_r} \tag{4.110}$$

$$C_n = C_{n_\beta} \cdot \beta + C_{n_{\delta R}} \cdot \delta R + \frac{pb}{2V_B} C_{n_p} + \frac{rb}{2V_B} C_{n_r} \tag{4.111}$$

The forces and moments for the vertical tail can then be expressed in the following equations.

$$(\Delta Y_a)_{VT} = C_y \, Sq \left(\frac{q_{VT}}{q}\right) \tag{4.112}$$

$$(\Delta L_a)_{VT} = C_\ell \, bSq \left(\frac{q_{VT}}{q}\right) \tag{4.113}$$

$$(\Delta N_a)_{VT} = C_n \, bSq \left(\frac{q_{VT}}{q}\right) \tag{4.114}$$

This force and these moments will be included in the total aerodynamic forces and moments.

Horizontal Stabilizer and Elevator. The aerodynamic forces and moments for the horizontal stabilizer will be defined in a similar manner as for the XC-142A. Elevator movement is considered in the z-force, and the pitching moment. The angle of attack of the tail α_t will be a function of interference damping terms, and considerations of downwash. The lift and drag coefficients of the tail can be expressed in the following relations.

$$C_{L_t} = C_{L_{\alpha_t}} \cdot \alpha_t \tag{4.115}$$

$$C_{D_t} = C_{L_t} + K \tag{4.116}$$

The horizontal stabilizer (HS) can contribute forces in the x and z directions and a pitching moment. The equations are as follows:

$$(\Delta X_a)_{HS} = [-C_{D_t} \cos (i_t - \alpha_t) - C_{L_t} \sin (i_t - \alpha_t)] \, Sq(\frac{q_{HS}}{q}) \tag{4.117}$$

$$(\Delta Z_a)_{HS} = [+C_{D_t} \sin (i_t - \alpha_t) - C_{L_t} \cos (i_t - \alpha_t)] \, Sq(\frac{q_{HS}}{q})$$
$$+ qScC_{m_{\delta E}} \cdot \delta E \tag{4.118}$$

$$(\Delta M_a)_{HS} = -(\Delta X_a)_{HS} \cdot h_{HS} + (\Delta Z_a)_{HS} \cdot \ell_{HS} + \frac{c^2 S \rho}{4} [C_{m_{q_1}} \cdot q_1 V + C_{m_{\dot{a}}} \cdot \dot{W}]$$
$$+ qScC_{m_{\delta E}} \cdot \delta E \tag{4.119}$$

ℓ_{HS} is the distance from the aircraft c.g. to the aerodynamic center of the horizontal stabilizer and h_{HS} is the height of the a.c. above the c.g. i_t is the incidence angle of the tail. Equations (4.117) to (4.119) will be included in the total aerodynamic force and moment expressions.

Fuselage. In a very direct way we can write the effects of the fuselage (F) on the total aerodynamic forces and moments. They are as follows:

$$(\Delta X_a)_F = -q \, S \, C_{D_o} \tag{4.120}$$

$$(\Delta Y_a)_F = + q \, S \, C_{y_\beta} \cdot \beta \tag{4.121}$$

$$(\Delta Z_a)_F = - q \, S \, C_{L_\alpha} \cdot \alpha \tag{4.122}$$

$$(\Delta M_a)_F = q\, Sc(C_{m_o} + C_{m_\alpha} \cdot \alpha) \tag{4.123}$$

$$(\Delta N_a)_F = q\, Sb\, C_{n_\beta} \cdot \beta \tag{4.124}$$

Equations (4.120) to (4.124) will be included in the total aerodynamic forces and moments.

Reaction Controls. The control vanes (horizontal (VH) and vertical (VV) are located in the exhaust flow of the jet engine. The reaction control (RC) vanes provide longitudinal and lateral control in the hover mode. The following coefficients are defined as in Reference 8.

$$C_{L_{VH}} = K_{VH}$$

$$C_{L_{VV}} = K_{VV}$$

$$C_{D_{VH}} = K\, C^2_{L_{VH}}$$

$$C_{D_{VV}} = K\, C^2_{L_{VV}}$$

$T_r = \dot{m}\,(V_E - U)$ -- residual thrust (T_r)

$$C_{m_{RC}} = \frac{\bar{x}_{VH}}{c_v} C_{L_{VH}} - \frac{\bar{z}_{VH}}{c_v} C_{D_{VH}} \tag{4.125}$$

$$C_{n_{RC}} = \frac{\bar{x}_{VV}}{b_v} C_{L_{VV}} - \frac{\bar{y}_{VV}}{b_v} C_{D_{VV}} \tag{4.126}$$

$\bar{x}_{VH}$, $\bar{x}_{VV}$, and $\bar{y}_{VV}$ are moment arms to control vanes.

$$(C_x) = -\,[C_{D_{VH}} + C_{D_{VV}}] \tag{4.127}$$

$$(C_y)_{RC} = -\,C_{L_{VV}} \tag{4.128}$$

$$(C_z)_{RC} = -\,C_{L_{VH}} \tag{4.129}$$

The force and moment equations then follow from equations (4.125) to (4.129) as:

$$(\Delta X_a)_{RC} = q_E\, S_V (C_x)_{RC} + T_r \qquad \left[\rho_E = .000776 \text{ slugs/ft}^3\right] \tag{4.130}$$

$$(\Delta Y_a)_{RC} = q_E\, S_V (C_y)_{RC} \qquad \left[V_F = 354.6 \text{ FPS}\right] \tag{4.131}$$

$$(\Delta Z_a)_{RC} = q_E\, S_V (C_z)_{RC} \qquad \left[q_E = \tfrac{1}{2}\rho_F\, V_E^2\right] \tag{4.132}$$

$$(\Delta M_a)_{RC} = q_E\, S_V\, c_v (C_m)_{RC} \tag{4.133}$$

$$(\Delta N_a)_{RC} = q_E\, S_V b_v (C_n)_{RC} \tag{4.134}$$

Equations (4.130) to (4.134) will be included in the total aerodynamic forces and moments.

EQUATIONS OF MOTION VZ-4DA. The aerodynamic force and moment terms developed for the VZ-4DA for each of the major aircraft components will be combined and set equal to their corresponding internal force and moment expressions developed in axis system section for the VZ-4DA.

Total Aerodynamic Forces and Moments. These expressions are as follows:

$$X_a = (\Delta X_a)_W + (\Delta X_a)_D + (\Delta X_a)_{HS} + (\Delta X_a)_F + (\Delta X_a)_{RC}$$

$$Y_a = (\Delta Y_a)_W + (\Delta Y_a)_D + (\Delta Y_a)_{VT} + (\Delta Y_a)_F + (\Delta Y_a)_{RC}$$

$$Z_a = (\Delta Z_a)_W + (\Delta Z_a)_D + (\Delta Z_a)_{HS} + (\Delta Z_a)_F + (\Delta Z_a)_{RC}$$

$$L_a = (\Delta L_a)_W + (\Delta L_a)_D + (\Delta L_a)_{VT}$$

$$M_a = (\Delta M_a)_W + (\Delta M_a)_D = (\Delta M_a)_{HS} + (\Delta M_a)_F + (\Delta M_a)_{RC}$$

$$N_a = (\Delta N_a)_W + (\Delta N_a)_D + (\Delta N_a)_{VT} + (\Delta N_a)_F + (\Delta N_a)_{RC}$$

Equations of Motion.- These equations are subject to simplification in the interference region and in the representation of duct fan forces and moments when further flight test data is available for the determination of typical magnitudes of terms.

1. X Force Equation

$$[-C_{D_t} \cos (i_t - \alpha_t) - C_{L_t} \sin (i_t \; (i_t - \alpha_t) \; Sq(\frac{q_{HS}}{q})$$

$$+ (C_x)_w \; S_{q_w} + K_3(\bar{C}_x)_D - q \; S \; C_{D_o} + q_E \; S_V(C_x)_{RC} + T_r$$

$$= m(\dot{U} + Wq_1 - Vr) + mg \sin \theta$$

2. Y Force Equation

$$(C_y)_w \; S_{q_w} + K_3(\bar{C}_y)_D \cdot C_y S_q(\frac{q_{VT}}{q}) + q \; S \; C_{y_\beta} \cdot \beta + q_E \; S_V(C_y)_{RC}$$

$$= m(\dot{V} + Ur - Wp) - mg \cos \theta \sin \Phi$$

3. Z Force Equation

$$[C_{D_t} \sin (i_t - \alpha_t) - C_{L_t} \cos (i - \alpha_t)] \; Sq(\frac{q_{HS}}{q})$$

$$+ (C_z) \; S_{q_w} + K_3(\bar{C}_z)_D$$

$$+ q \; Sc \; C_{m_{\delta E}} \cdot \delta E - qS \; C_{L_\alpha} \cdot \alpha + q_E \; S_V(C_z)_{RC}$$

$$= m(\dot{W} + Vp - Uq_1) - mg \cos \theta \cos \Phi$$

4. Roll Equation

$$(C_\ell)_w \; bSq_w + K_3 \; b(\bar{C}_\ell)_D + C_\ell \; b \; Sq(\frac{q_{VT}}{q})$$

$$= I_{11}\dot{p} - I_{13} \; (\dot{r} + pq_1) + (I_{33} - I_{22}) \; q_1 r$$

$$+ (I_E \dot{\Omega}_E) \cos (i_w + \delta_D) - q_1 \; (I_E \Omega_E) \sin (i_w + \delta_D)$$

5. Pitch Equation

$$(C_m)_w \; c \; Sq_w + K_3 c(\bar{C}_m)_D + q \; Sc(C_{m_o} + C_{m_\alpha} \cdot \alpha) + q_E \; S_V \; c_V(C_m)_{RC}$$

$$-(\Delta X_a)_{HS} \cdot h_{HS} + (\Delta Z_a)_{HS} \cdot \ell_{HS} + \frac{c^2 S\rho}{4} [C_{m_{q_1}} \cdot q_1 V + C_{m_{\dot{a}}} \cdot \dot{W}]$$

$$+ qSc \; C_{m_{\delta E}} \cdot \delta E$$

$$= I_{22}\dot{q}_1 - I_{13}(r^2 + p^2) + (I_{11} - I_{33}) \; pr$$

$$+ p(I_E \Omega_E) \sin (i_w + \delta_D) + r(I_E \Omega_E) \cos (i_w + \delta_D)$$

6. Yaw Equation

$$(C_n)_w \, bSq_w + K_3 b(\bar{C}\ )_D + C_n \, bSq\left(\frac{q_{VT}}{q}\right) + q\, Sb\, C_{n_\beta} \cdot \beta$$

$$+ q_E \, S_V \, b_V (C_n)_{RC}$$

$$= I_{33}\dot{r} - I_{13}\,(\dot{p} - q_1 r) + I_{22} - I_{11})\, pq_1$$

$$- (I_E \dot{\Omega}_E) \sin (i_w + \delta_D) - q_1\, (I_E \Omega_E) \cos (i_w + \delta_D)$$

No simplifications have been made on the right side of these six equations.

TILT PROPELLER

The Curtiss-Wright X-19 V/STOL aircraft will be used as the example in the development of tilt propeller (tilt prop) V/STOL simulation equations of motion. The development which follows is based upon the type of arguments presented for the tilt-wing and information gleaned from Reference 4.

In order to gain some idea of the X-19 let us consider Figure 3 and Table 6. In Figure 3, a three-view arrangement of the X-19 is shown. Table 6 is a list of definition of symbols contained in the X-19 equations.

With this general idea of the X-19, we may now develop the mathematical model for the X-19. First we define, as necessary applicable axis systems. Second, develop aerodynamic coefficients. Finally, write the equations of motion for the X-19.

AXIS SYSTEMS FOR X-19. It appears that the forward and aft wing chord are parallel to the x-body axis of the aircraft. In this aircraft the aft wing functions as the horizontal stabilizer with elevator. The total velocity of the wing, V_w, can be expressed easily in aircraft body axes. In addition separate axis systems to describe propeller velocities are not necessary since the propeller velocities can be expressed in terms of aircraft body axis velocities (U, V, W). Consequently, no additional axis systems will be necessary other than the conventional inertial, body, stability or wind axes.

In addition, there will be no gyroscopic effects from the rotating masses of the propellers and their drive shafts. The four propellers are mechanically linked to rotate at the same speed and the starboard propellers turn in the opposite direction of the port propellers. If one or more propellers failed there would be gyroscopic effects.

Symbol	Definition
F	Fuselage
w	Wing
n	Main propellers n = 1, 2, Front. n = 3, 4 Aft. Top view, left to right
vt	Vertical Tail
α_F	Angle of attack - fuselage
α_{wf}	Angle of attack - wing forward
α_{wa}	Angle of attack - wing aft
β_F	Sideslip angle - fuselage
β_{wf}	Sideslip angle - wing forward
β_{wa}	Sideslip angle - wing aft
i_p	Incidence angle - propeller nacelle
V_B	Total velocity in aircraft body axes
V_w	Total velocity in wind stability axes
c	Mean aerodynamic chord
b	Wing span
S	Total area of wing
q_{wf}	Dynamic pressure at forward wing due to power-on effects
q_{wa}	Dynamic pressure at aft wing due to power-on effects
q	Dynamic pressure - free stream
q_{vt}	Dynamic pressure at vertical tail
T	Total thrust of main propellers
J_n	Advance ratio main propeller
J'_n	Advance ratio normal to propeller disk
$N_n = N$	RPM - main propeller

Table 6. Definition of Symbols Used in X-19 Equations

Symbol	Definition
N_o	Nominal RPM - main propeller
B_n	Blade pitch angle - main propeller
T_n	Main propeller thrust
N_n	Main propeller thrust component normal to T_n.
M_n	Main propeller moment (initially pitching)
Q_n	Main propeller torque coefficient
D_n	Diameter main propeller

Table 6. Definition of Symbols Used in X-19 Equations (Cont'd.)

AERODYNAMIC FORCES AND MOMENTS - X-19. In order to develop expressions for X_a, Y_a, Z_a, L_a, M_a, and N_a, contributions from the major airframe components of the X-19 will be considered separately as was done for the tilt-wing aircraft. The major components to be considered are the propellers, the forward wing and aft wings, the vertical tail and rudder, and the fuselage.

Propeller. The inclination of the propeller nacelles from the x-body axis is defined by propeller shafts, angle of tilt, i_p. In Figure 3 we see the angle i_p and the geometry defining the total propeller velocity, V_n. For the velocity, V_n, n denotes the particular propeller (n = 1, 2, 3, 4). The front left propeller is number 1, the front right propeller is number 2, the rear left propeller is number 3 and the rear right propeller is number 4. Each propeller nacelle is located by x_n, y_n and z_n body axis components. Due to the symmetric placement of the propellers; $x_1 = x_2, -x_3 = -x_4, -y_1 = y_2, -y_3 = y_4, -z_1 = -z_2$ and $-z_3 = -z_4$. These body axes moment arms (s_n, y_n and z_n) are multiplied by the appropriate body axis angular velocity (p, q_1 and r) in order to give tangential velocity components of the propeller velocity components.

$$u_n = U - y_n(p \sin i_p + r \cos i_p) - q_1(x_n \sin i_p - z_n \cos i_p) \qquad (4.135)$$

$$v_n = V + x_n r \qquad (4.136)$$

$$w_n = W + y_n(p \cos i_p - r \sin i_p) - q_1(x_n \cos i_p + z_n \sin i_p) \qquad (4.137)$$

We will now define the propeller aerodynamic coeffients C_{T_n}, C_{P_n}, C_{N_n} and C_{M_n}. These coefficients are the same as those for the main propellers of the XC-142A, however there is no C_{Y_n} for the X-19, since the port propellers turn in an opposite direction to the starboard propellers. First let us define the advance ratio (J_n) for each propeller and the advance ratio normal to the propeller disk (J'_n)

$$J_n = \frac{60\, V_n}{N_n D},$$

$$J'_n = J_n \cos \psi_n \qquad (4.138)$$

The symbol N_n (the particular propeller RPM and since all propellers are mechanically linked to turn at the same rate) $N_n = N$. The number 60 changes RPM to RPS, D is the diameter of the propeller disk and B_n is defined as the blade pitch angle of the particular propeller. The aerodynamic coefficients can then be developed in terms of advance ratio and blade pitch. The angle ψ_n is the angle between the propeller thrust vector and v_n as shown in Figure 7. ψ_n is defined the same as for the tilt wing as is β_n--equations (4.1) and (4.2).

$$C_{T_n} = C_{T_o} + \frac{\partial C_T}{\partial J'} J'_n + \frac{\partial C_T}{\partial B} B_n + \frac{\partial^2 C_T}{\partial B \partial J'} B_n J'_n \tag{4.139}$$

$$C_{P_n} = C_{P_o} + \frac{\partial C_P}{\partial J'} J'_n + \frac{\partial C_P}{\partial B} B_n + \frac{\partial^2 C_P}{\partial B \partial J'} B_n J'_n \tag{4.140}$$

$$C_{N_n} = \frac{\partial}{\partial B} \left[\frac{\partial (C_n \cot \psi)}{\partial J'}\right] B_n J_n \sin \psi_n \tag{4.141}$$

$$C_{M_n} = \frac{\partial}{\partial J'} \left[\frac{\partial C_M}{\partial \psi_n}\right] \psi_n J'_n \tag{4.142}$$

C_{T_n}, C_{P_n}, C_{M_n} are defined similar to equations (4.7), (4.8), (4.9) and (4.11). Since blade pitch is the prime control parameter in the X-19, the second partial derivative $\frac{\partial 2}{\partial B \partial J'}$ is retained in equations (4.139) and (4.140) whereas $\frac{\partial^2}{\partial J'^2}$ is neglected. C_{T_n} is the coefficient of thrust (T_n), C_{P_n} is the coefficient of power used to express torque (Q_n), C_{N_n} is the coefficient of normal thrust (N_n) and C_{M_n} is the longitudinal propeller hub moment coefficient, (M_n).

Before expressing the force and moment contributions in body axes due to the propellers, we will define T_n, N_n, M_n and Q_n (Reference 6).

$$T_n = D^4 \left(\frac{N}{N_o}\right) \left(\frac{\rho}{\rho_o}\right) C_{T_n} \tag{4.143}$$

$$N_n = D^4 \left(\frac{N}{N_o}\right) \left(\frac{\rho}{\rho_o}\right) C_{N_n} \tag{4.144}$$

$$M_n = D^5 \left(\frac{N}{N_o}\right) \left(\frac{\rho}{\rho_o}\right) C_{M_n} \tag{4.145}$$

$$Q_n = \frac{D^5}{2} \left(\frac{N}{N_o}\right) \left(\frac{\rho}{\rho_o}\right) C_{P_n} \tag{4.146}$$

N_o is the maximum RPM of the propellers and ρ_o is the air density at sea level on a standard day.

From equations (4.143) through (4.146) we can then write the propeller force and moment expressions in a manner similar to equations (4.17) through (4.22) of the XC-142A.

$$(\Delta X_a)_p = \sum_{n=1}^{4} (T_n \cos i_p - N_n \cos \beta_n \sin i_p) \tag{4.147}$$

$$(\Delta Y_a)_p = \sum_{n=1}^{4} (-N_n \sin \beta_n) \tag{4.148}$$

$$(\Delta Z_a)_p = \sum_{n=1}^{4} (-T_n \sin i_p - N_n \cos \beta_n \cos i_p) \tag{4.149}$$

$$\begin{aligned}(\Delta L_a)_p = &- [(\Delta Z_a)_{p_1} - (\Delta Z_a)_{p_2}]y_1 - [(\Delta Z_a)_{p_3} - (\Delta Z_a)_{p_4}]y_3 \\ &+ [(\Delta Y_a)_{p_1} + (\Delta Y_a)_{p_2}]z_1 + [(\Delta Y_a)_{p_3} + (\Delta Y_a)_{p_4}]z_3 \\ &- \sum_{n=1}^{4} M_n \sin \beta_n \sin i_p\end{aligned} \tag{4.150}$$

$$\begin{aligned}(\Delta M_a)_p = {} & x_1(T_1 + T_2) \sin i_p - x_3(T_3 + T_4) \sin i_p \\ &- z_1(T_1 + T_2) \cos i_p - z_3(T_3 + T_4) \cos i_p \\ &+ z_1(N_1 \cos \beta_1 + N_2 \cos \beta_2) \sin i_p \\ &+ z_3(N_3 \cos \beta_3 + N_4 \cos \beta_4) \sin i_p \\ &+ x_1(N_1 \cos \beta_1 + N_2 \cos \beta_2) \cos i_p\end{aligned} \tag{4.151}$$

$$- x_3(N_3 \cos \beta_3 + N_4 \cos \beta_4) \cos i_p$$

$$+ \sum_{n=1}^{4} M_n \cos \beta_n$$

$$(\Delta N_a)_p = [(X_a)_{p_1} - (X_a)_{p_2}] y_1 + [(X_a)_{p_3} - (X_a)_{p_4}] y_3$$

$$+ [(Y_a)_{p_1} + (Y_a)_{p_2}] x_1 - [(Y_a)_{p_3} + (Y_a)_{p_4}] x_3 \qquad (4.152)$$

$$- \sum_{n=1}^{4} M_n \sin \beta_n \cos i_p$$

Note that x_1, x_3, y_1, y_3, z_1 and z_3 are positive values. Th propeller force and moment equations (4.147) through (4.152) will be included in the total aerodynamic forces and moments.

Forward Wing. The forward wing forces and moments are developed as for the XC-142A in Section IV. It is assumed that the wing incidence angle, i_w, is zero and that the induced velocity, ΔV, is defined as equation (4.34) when i_p is zero. ΔV will change as a function cos i_p and flap position when the aircraft is in hover and transition regions. Consequently, as a preliminary estimate, the wing velocity u_{wf} is as follows:

$$u_{wf} = u_p + (\Delta V)_{TP} \qquad (4.153)$$

where $(\Delta V)_{TP} = K \, \Delta V \cos i_p$

For forward flight where i_p equals zero then u_p equals V_B and $K = 1$. K is a function of flaps and interference effects as the propeller wash hits the wing at high i_p angles. K and cos i_p will tend to decrease ΔV as i_p increases. As in equation (4.36) q_{wf} is defined similarly as:

$$q_{wf} = (q + 2\frac{(T_1 + T_2)}{2\pi D^2}) \qquad (4.154)$$

The forward wing aerodynamic coefficients are C_D, C_L, $(C_l)_{wf}$, $(C_m)_{wf}$ and $(C_n)_{wf}$ C_D and C_L define $(C_x)_{wf}$ and $(C_z)_{wf}$ respectively.

Since the development of these coefficients is in wing axes, rolling and turning rates necessary to define these coefficients are transformed from body axes. The only difference is in the angle of attack of the wing, α_{wf}, which is from equation (4.32) as follows:

$$\alpha_{wf} = \tan^{-1} \left[\frac{w_p}{u_p + (\Delta V)_{TP}} \right] \tag{4.155}$$

where $w_{wf} = w_p$; $W = u_p \sin i_p$

and $u_{wf} = u_p + (\Delta V)_{TP}$; $U = u_p \cos i_p$

In a similar manner β_{wf} is defined as:

$$\beta_{wf} = \tan^{-1} \frac{V}{V_{wf}} \tag{4.156}$$

where $v_p = v_{wf} = V$

and $V_{wf} = [W^2 + (U + \Delta V)_{TP})^2 + V^2]^{1/2}$

We write for the wing rolling rate (p_{wf}) and the wing turning rate (r_{wf}) the following equations:

$$p_{wf} = p \cos \alpha_{wf} + r \sin \alpha_{wf} \tag{4.157}$$

$$r_{wf} = -p \sin \alpha_{wf} + r \cos \alpha_{wf} \tag{4.158}$$

These are just equations (4.37) and (4.38) where $\xi = -\alpha_w$ since $i_w = 0$ in $\xi = (i_w - \alpha_{wf})$.

Following the tilt wing the aerodynamic coefficients for the wing are as follows:

$$C_D = C_{D_o} + C_{D_{\delta F}} \cdot \delta F \tag{4.159}$$

$$C_L = C_{L_o} + C_{L_{\delta F}} \cdot \delta F + C_{L_{\alpha_F}} \cdot \alpha_{wf} \tag{4.160}$$

C_D and C_L do not reflect as strong a flap dependence as in the XC-142A. This is a preliminary estimate of C_D and C_L which may be modified by further manufacturer data. $(C_\ell)_{wf}$ $(C_m)_{wf}$ and $(C_n)_{wf}$ will be defined as for the XC-142A.

$$(C_\ell)_{wf} = C_{\ell_{\beta_F}} \cdot \beta_{wf} + C_{\ell_{\delta_A}} \cdot \delta A + \frac{b}{2V_{wf}} C_{\ell_p} \cdot p_{wf} + \frac{b}{2V_{wf}} C_{\ell_r} \cdot r_{wf} \quad (4.161)$$

$$(C_m)_{wf} = C_{m_o} + C_{m_{\delta F}} \cdot \delta F + \frac{c}{2V_{wf}} C_{m_{q_1}} \cdot q_1 \quad (4.162)$$

$$(C_n)_{wf} = C_{n_{\beta F}} \cdot \beta_{wf} + C_{n_{\delta A}} \cdot \delta A + \frac{b}{2V_{wf}} C_{n_p} \cdot p_{wf} + \frac{b}{2V_{wf}} \cdot C_{n_r} \cdot r_{wf} \quad (4.163)$$

Equations (4.159) through (4.163) will now be written in body axes by rotating through an angle $(-\alpha_w)$.

$$(C_x)_{wf} = - C_D \cos \alpha_{wf} + C_L \sin \alpha_{wf} \quad (4.164)$$

$$(C_z)_{wf} = - C_D \sin \alpha_{wf} - C_L \cos \alpha_{wf} \quad (4.165)$$

$$(C_\ell)_{wf} = (C_\ell)_{wf} \cos \alpha_{wf} - (C_n)_{wf} \sin \alpha_{wf} \quad (4.166)$$

$$(C_m)_{wf} = (C_m)_{wf} - \frac{Z_{ac}}{c} (C_x)_{wf} - \frac{X_{ac}}{c} (C_z)_{wf} \quad (4.167)$$

$$(C_n)_{wf} = (C_\ell)_{wf} \sin \alpha_{wf} + (C_n)_{wf} \cos \alpha_{wf} \quad (4.168)$$

X_{ac} and Z_{ac} are the respective distance from the c.g. of the aircraft in body axes to the aerodynamic center (a.c.) of the forward wing in the x-z plane. c is the mean aerodynamic chord of the forward wing.

The wing force and moment contributions can then be written from the coefficients expressed in equations (4.164) through (4.168). S is the forward wing area and q_{wf} the dynamic pressure at the wing.

$$(\Delta X_a)_{wf} = (C_x)_{wf} \, S \, q_{wf} \quad (4.169)$$

$$(\Delta Z_a)_{wf} = (C_z)_{wf} \, S \, q_{wf} \quad (4.170)$$

$$(\Delta L_a)_{wf} = (C_\ell)_{wf} \, bS \, q_{wf} \quad (4.171)$$

$$(\Delta M_a)_{wf} = (C_m)_{wf} \, cS \, q_{wf} \quad (4.172)$$

$$(\Delta N_a)_{wf} = (C_n)_{wf} \, bS \, q_{wf} \quad (4.173)$$

Vertical Stabilizer and Rudder. The discussion is the same as for the XC-142A since there is obvious similarity in configuration of the vertical stabilizer for the propeller wash effects on the vertical stabilizer for the XC-142A and the X-19. Forces and moments for vertical tail (vt) and rudder arise from the relative wind pushing against the vertical tail surfaces thereby causing a turning moment due to control input to the rudder. This gives side force, as well as rolling and turning moments.

Wind pushing against the vertical stabilizer and rudder yields a side force $(\Delta Y_a)_{vt}$ and rolling moment $(\Delta L_a)_{vt}$ respectively which are non-dimensionalized in terms of aerodynamic coefficients as C_y and C_ℓ. The rolling moment is coupled in rolling velocity p and turning velocity r. We can then write for C_y and C_ℓ the following expressions.

$$C_y = C_{y_\beta} \cdot \beta + C_{y_{\delta R}} \cdot \delta R \tag{4.174}$$

C_{y_β} is the change in side force coefficient with changing sideslip angle. It acts as a damping term. $C_{y_{\delta R}}$ is the change in side force coefficient with rudder deflection and represents the controllable term in the expression. This is important in the use of Automatic Stabilization Equipment (ASE).

$$C_\ell = C_{\ell_\beta} \cdot \beta + C_{\ell_{\delta R}} \cdot \delta R + \frac{pb}{2V_B} C_{\ell_p} + \frac{rb}{2V_B} C_{\ell_r} \tag{4.175}$$

C_{ℓ_β} is the change in rolling moment with variation in sideslip angle. The vertical tail is normally above the X axis, hence a side force on the tail gives a rolling moment. $C_{\ell_{\delta R}}$ is the change in rolling moment coefficient due to rudder deflection. C_{ℓ_p} is the roll damping derivative. C_{ℓ_r} is the change in rolling moment coefficient with change in yawing velocity.

Rudder deflection will give a turning moment $(\Delta N_a)_{vt}$ which can be expressed by the aerodynamic coefficient C_n.

$$C_n = C_{n_\beta} \cdot \beta + C_{n_{\delta R}} \cdot \delta R + \frac{pb}{2V_B} C_{n_p} + \frac{rb}{2V_B} C_{n_r} \tag{4.176}$$

C_{n_β} is the weathercock or static directional derivative. $C_{n_{\delta R}}$ is rudder effectiveness, the change in yawing moment coefficient with rudder deflection. C_{n_p} is the change in yawing moment coefficient with varying rolling velocity. C_{n_r} is the yaw damping derivative. The tail is the main contribution to C_{n_r}.

The forces and moments for the vertical tail can then be expressed in the following equations:

$$(\Delta Y_a)_{vt} = C_y \, S_q\left(\frac{q_{vt}}{q}\right) \quad (4.177)$$

$$(\Delta L_a)_{vt} = C_l \, bSq\left(\frac{q_{vt}}{q}\right) \quad (4.178)$$

$$(\Delta N_a)_{vt} = C_n \, bSq\left(\frac{q_{vt}}{q}\right) \quad (4.179)$$

Here q is the free stream dynamic pressure and q_{vt} is the vertical tail dynamic pressure. These force and moments will be included in the total aerodynamic forces and moments.

Aft Wing. The aft wing (wa) has two propellers placed in a similar configuration as the forward wing and acts as a combination wing and horizontal stabilizer. When the four propellers are simultaneously rotated by the angle, i_p, the aft wing is either in or out of the front propeller wash. If the aft wing is out of the front propeller wash, the aft wing will be practically a duplicate of the forward wing in aerodynamics. If the aft wing is in the front propeller wash, the aft wing has induced velocity effects from its own and the front propellers. The description of downwash, as for the XC-142A Horizontal Stabilizer is not applicable to the X-19. However, since the aft wing has elevator control and no aileron control, the forces and moments developed will lie in the x-z body axes plane as for a horizontal stabilizer.

As for the forward wing, the aft wing incidence angle is assumed to be zero. Induced velocity effects for the aft wing are as follows:

$$u_{wa} = u_p + (\Delta V)_{TP} \quad (4.180)$$

$$q_{wa} = \left(q + \frac{T_1 + T_2 + T_3 + T_4}{\pi D^2}\right) \quad (4.181)$$

Equation (4.180) is the same as (4.153), but is for the aft wing out of front propeller wash. V_{wa} is the same as V_{wf} out of wash, and V_{wa} differs from V_{wf} by the effect of induced velocity in wash effects. For the aft wing in front propeller wash, equation (4.180) becomes:

$$u_{wa} = u_p + 3/2(\Delta V)_{TP} \quad (4.182)$$

where $(\Delta V)_{TP} = K_2 \, \Delta V \cos i_p$

Notice that if $i_p = 0$, $(\Delta V)_{TP} = \frac{3K_2}{2} \Delta V = \frac{3\Delta V}{2}$ ($K_2 \approx 1$ at $i_p = 0$)

K_2 accounts for interference effects. From momentum theory the induced velocity, ΔV, does not double on the aft wing from the effect of two propellers blowing on the wing in parallel but ΔV increases by one-half. The coefficients $(C_D)_{wa}$, $(C_L)_{wa}$ and $(C_m)_{wa}$ are defined similar to equations (4.159), (4.160) and (4.162), but using elevator controls instead of flap controls.

$$(C_D)_{wa} = C_{D_o} + C_{D_{\delta E}} \cdot \delta E \quad (4.183)$$

$$(C_L)_{wa} = C_{L_o} + C_{L_{\delta E}} \cdot \delta E + C_{L_{\alpha_F}} \cdot \alpha_{wa} \quad (4.184)$$

$$(C_m)_{wa} = C_{m_o} + \frac{c}{2V_{wa}} C_{m_{q_1}} \cdot q_1 + C_{m_{\dot{\alpha}}} \cdot \dot{W} \quad (4.185)$$

The coefficients $(C_D)_{wa}$, $(C_L)_{wa}$ and $(C_m)_{wa}$ are rotated to body axes by the aft wing angle of attack (α_{wa}). The coefficients in body axes are similar to equations (4.164), (4.165) and (4.167).

$$(C_x)_{wa} = -(C_D)_{wa} \cos \alpha_{wa} + (C_L)_{wa} \sin \alpha_{wa} \quad (4.186)$$

$$(C_z)_{wa} = -(C_D)_{wa} \sin \alpha_{wa} - (C_L)_{wa} \cos \alpha_{wa} \quad (4.187)$$

From the coefficients defined in equations (4.185), (4.186) and (4.187) the force and moment contributions of the aft wing are as follows:

$$(\Delta X_a)_{wa} = (C_x)_{wa} \, S \, q_{wa} \quad (4.188)$$

$$(\Delta Z_a)_{wa} = (C_z)_{wa} \, S \, q_{wa} \quad (4.189)$$

$$(\Delta M_a)_{wa} = (C_m)_{wa} - \frac{Z_{ac}}{c} (\Delta X_a)_{wa} + \frac{X_{ac}}{c} (\Delta Z_a)_{wa} \quad (4.190)$$

In equation (4.190), X_{ac} and Z_{ac} are the respective distances from the aircraft c.g. to the aerodynamic center (ac) of the aft wing in the x-z plane. c is the mean aerodynamic chord of the aft wing.

Fuselage. In a very direct manner we can write the effects of the fuselage (F) on the total aerodynamic forces and moments as for the XC-142A.

We have for the forces

$$(\Delta X_a)_F = -\frac{1}{2}\rho V_B^2 S C_{D_o} \tag{4.191}$$

C_{D_o} is the equilibrium drag coefficient.

$$(\Delta Y_a)_F = +\frac{1}{2}\rho V_B^2 S C_{y_{\beta_F}} \cdot \beta_F \tag{4.192}$$

$C_{y_{\beta_F}}$ is the change in side force with respect to a changing sideslip angle.

$$(\Delta Z_a)_F = -\frac{1}{2}\rho V_B^2 S C_{L_{\alpha_F}} \cdot \alpha_F \tag{4.193}$$

$C_{L_{\alpha_F}}$ is the change in lift coefficient with varying angle of attack. This is also known as the lift curve slope.

We have for the moments:

$$(\Delta L_a)_F = 0$$

$$(\Delta M_a)_F = \frac{1}{2}\rho V_B^2 S c C_{m_o} + \frac{1}{2}\rho V_B^2 S c C_{m_{\alpha_F}} \cdot \alpha_F \tag{4.194}$$

C_{m_o} is the aerodynamic pitching moment coefficient in equilibrium flight and $C_{m_{\alpha_F}}$ is the longitudinal static stability derivative.

$$(\Delta N_a)_F = \frac{1}{2}\rho V_B^2 S b C_{n_{\beta_F}} \cdot \beta_F \tag{4.195}$$

$C_{n_{\beta_F}}$ is the static directional or "weathercock" derivative.

In the above expressions $q = \frac{1}{2}\rho V_B^2$ which is the dynamic pressure, b is the forward wing span, c is the mean aerodynamic chord and S is the forward wing area.

For hovering and in transition regions β_F is assumed small so that $\cos \beta_F = 1$.

EQUATIONS OF MOTION - X-19. The aerodynamic force and moment terms developed in this section for each of the major aircraft components of the X-19 will be combined and finally expressed in the equations of motion.

Total Aerodynamic Forces and Moments. The total forces and moments are as follows:

$$X_a = (\Delta X_a)_p + (\Delta X_a)_{wf} + (\Delta X_a)_{wa} + (\Delta X_a)_F$$

$$Y_a = (\Delta Y_a)_p + (\Delta Y_a)_{vt} + (\Delta Y_a)_F$$

$$Z_a = (\Delta Z_a)_p + (\Delta Z_a)_{wf} + (\Delta Z_a)_{wa} + (\Delta Z_a)_F$$

$$L_a = (\Delta L_a)_p + (\Delta L_a)_{wf} + (\Delta L_a)_{vt}$$

$$M_a = (\Delta M_a)_p + (\Delta M_a)_{wf} + (\Delta M_a)_{wa} + (\Delta M_a)_F$$

$$N_a = (\Delta N_a)_p + (\Delta N_a)_{wf} + (\Delta N_a)_{vt} + (\Delta N_a)_F$$

Equations of Motion Expanded. The forces and moments are expressed in body axes. These equations are subject to revision when further data are available concerning the X-19.

1. X Force Equation.

$$\sum_{n=1}^{4} (T_n \cos i_p - N_n \cos \beta_n \sin i_p) + (C_x)_{wf} Sq_{wf}$$

$$+ (C_x)_{wa} Sq_{wa} - C_{D_o} Sq$$

$$= m (\dot{U} + W_{q_1} - Vr) + mg \sin \theta$$

2. Y Force Equation.

$$\sum_{n=1}^{4} (-N_n \sin \beta_n) + C_y Sq_{vt} + Sq\, C_{y_{\beta_F}} \cdot \beta_F$$

$$= m(\dot{V} + Ur - Wp) - mg \cos \theta \sin \Phi$$

3. Z Force Equation.

$$\sum_{n=1}^{4} (-T_n \sin i_p - N_n \cos \beta_n \cos i_p) + (C_z)_{wf} Sq_{wf}$$

$$+ (C_z)_{wa} Sq_{wf} - Sq\, C_{L_{\alpha_F}} \cdot \alpha F$$

$$= m(\dot{W} + Vp - Uq_1) - mg \cos \theta \cos \Phi$$

4. Roll Equation.

$$-[(T_1 - T_2) \sin i_p + (N_1 \cos \beta_1 - N_2 \cos \beta_2) \cos i_p] y_1$$

$$+[(T_3 - T_4) \sin i_p + (N_3 \cos \beta_3 - N_4 \cos \beta_4) \cos i_p] y_3$$

$$-[N_1 \sin \beta_1 + N_2 \sin \beta_2] z_1 - N_3 \sin \beta_3 + N_4 \sin \beta_4] z_3$$

$$- \sum_{n=1}^{4} M_n \sin \beta_n \sin i_p + (C\ )_{wf}\, bSq_{wf} + C\, bSq_{vt}$$

$$+ I_{13} (\dot{r} + pq_1) - (I_{33} - I_{22})\, q_1 r$$

$$= I_{11}\dot{p}$$

5. Pitch Equation.

$$[x_1(T_1 + T_2) - x_3(T_3 + T_4)] \sin i_p - [z_1(T_1+T_2) + z_3(T_3+T_4)(] \cos i_p$$

$$+[z_1(N_1 \cos \beta_1 + N_2 \cos \beta_2) + z_3(N_3 \cos \beta_3 + N_4 \cos \beta_4)] \sin i_p$$

$$+[x_1(N_1 \cos \beta_1 + N_2 \cos \beta_2) - x_3(N_3 \cos \beta_3 + N_4 \cos \beta_4)] \cos i_p$$

$$+ \sum_{n=1}^{4} M_n \cos \beta_n + (C_m)_{wf}\, cSq_{wf} + (C_m)_{wf}\, cSq_{wf} + qcS(C_{m_o} + C_{m_{\alpha_F}} \cdot \alpha_F)$$

$$+ I_{13} (r^2 - p^2) - (I_{11} - I_{33})\, pr$$

$$= I_{22}\dot{q}_1$$

c. Yaw Equation.

$$[(T_1 - T_2) \cos i_p + (N_2 \cos \beta_2 - N_1 \cos \beta_1) \sin i_p]y_1$$

$$[(T_3 - T_4) \cos i_p + (N_4 \cos \beta_4 - N_3 \cos \beta_3) \sin i_p]y_3$$

$$+ (N_3 \sin \beta_3 + N_4 \sin \beta_4) x_3 - (N_1 \sin \beta_1 + N_2 \sin \beta_2) x_1$$

$$- \sum_{n=1}^{4} M_n \sin \beta_n \cos i_p + (C_n)_{wf} \, bSq_{wf} + C_n \, bSq_{vt} + qBSC_{n_{\beta_F}} \cdot \beta_F$$

$$+ I_{13}(\dot{p} - q_1 r) - (I_{22} - I_{11}) rq_1$$

$$= I_{33}\dot{r}$$

External stores, rough air or landing gear conditions are not included in these equations.

FAN-IN-WING

The General Electric/Ryan Aeronautical XV-5A V/STOL airplane will be used as the example in the development of fan-in-wing V/STOL simulation equations of motion. The equations of Section III will be directly applicable to the simulation equations for this aircraft.

In order to obtain some idea of the XV-5A consider Figure 4 and Table 7. In Figure 4, a three-view arrangement of the XV-4A is shown. Table 7 contains some of the physical characteristics of the aircraft.

With this general idea of the XV-5A, we may now develop the mathematical model for the XV-5A. First, we will define, if necessary, applicable axis systems. Second, we develop additional aerodynamic coefficients. Finally, we write the equations of motion for XV-5A.

AXIS SYSTEMS FOR THE XV-5A. The XV-5A has no change in its physical configuration during the hover, transition and normal flight. The only gross change from hover to normal flight is from thrust generated from the nose and wing fans to thrust from the tail pipes. The thrust developed by the fans during hover and transition should be available as a function of fan revolutions per second (rps) and fan louver position which directs the thrust of the fans. The thrust of the aircraft is described sufficiently in the axis systems used for ordinary jet aircraft simulation (Reference 2). Consequently, no axis systems in addition to the inertial, body, stability or wind axes will be necessary, to develop equations of motion for the XV-5A.

In addition, there will be no gyroscopic effects from the wing fans since they remain in the same geometric plane, are free wheeling in opposite directions and are driven from a common gas source at approximately the same rps. The fan in the nose, also free wheeling, does not give any appreciable lift, but is used as a pitch control. The doors in the fuselage ventral to the nose fan are linked to the control stick in order to control the flow of air through the nose fan thus giving pitch control.

$I_{NF}\ \Omega_{NF}\ q_1$, L_a term.

$I_{NF}\ \Omega_{NF}\ p$, M_a term.

$I_{NF}\ \dot{\Omega}_{NF}$, N_a term.

It may be determined after consideration of data from the manufacturer that these terms will be negligible, but for the present they will be included in the equations of motion.

Weight	
Empty	9,200 lbs.
Design Gross	12,000 lbs.
Overload Ferry (Conventional Take off)	13,600 lbs.
Dimensions	
Wing Span	29.8 ft.
Fuselage Length (over-all)	44.5 ft.
Height	14.7 ft.
Powerplant	
Lift Fan System - G.E. X353-5B	
Fan Diameter, Installed	76 inches
Fan Thickness	14.5 inches
Turbojet - G.E. J85-5	
Over-all Length with Diverter Valve	67.2 inches
Maximum Diameter	17.7 inches

Table 7. Selected Physical Characteristics - XV-5A

AERODYNAMIC FORCES AND MOMENTS XV-5A. It is proposed that since the XV-5A has no special problems in development of the equations of motions that equations such as (3.7) through (3.12) or those by Connelly, in Reference 2, be used for simulation purposes. In equations 3.7 and 3.9 the engine thrust will be defined to conform with the fan-in-wing.

The total engine thrust, T, is composed of either fan thrust, T_F, or jet thrust T_j.

$$T_F = T_{F_1} + T_{F_2} \qquad \begin{array}{l} 1 \text{ is port fan} \\ 2 \text{ is starboard fan} \end{array} \qquad (4\text{-}196)$$

Due to cockpit control of the diverter valve, T is equal to T_F or T_j. For example, when the aircraft has enough forward speed to be supported on wing lift the diverter valve is moved so that all the incoming air passes through the jet engine for forward thrust (i.e. T_j) and the fan inlet and exits are closed. There is a three to four second delay before the fan develops 100% lift when the diverter valve is switched from T_j to T_F. This delay is easily simulated and rids the thrust of a discontinuity between T_j and T_F.

Hover Flight XV-5A. The equations for hover flight are equations (3.7) through (3.12) with the aerodynamic coefficients considered negligible. We can then write the following expressions:

$$m(\dot{U} + Wq_1 - Vr) = T_J \cos \alpha_T + T_F \sin \theta_L \tag{4.197}$$

$$m(\dot{V} + Ur - Wp) = g \cos \theta \sin \Phi \tag{4.198}$$

$$m(\dot{W} + Vp - Uq_1) = -T_J \sin \alpha_T - T_F \cos \theta_L + f(T_{NF})$$

$$\dot{p} = \frac{I_{13}}{I_{11}} \dot{r} - \frac{I_{NF}\ \Omega_{NF}}{I_{11}} q_1 + \frac{\bar{y}}{I_{11}} (T_{F_1} \cos \theta_{L_1} - T_{F_2} \cos \theta_{L_2})$$

$$+ \frac{qSb}{I_{11}} C_{\ell_{\delta A}} \cdot \delta A \tag{4.199}$$

$$\dot{q}_1 = \frac{I_{NF}\ \Omega_{NF}}{I_{22}} p + \frac{\ell_{NF}\ T_{NF} \cos \theta_{NL}}{I_{22}} + \frac{qSc}{I_{22}} C_{m_{\delta E}} \cdot \delta E \tag{4.200}$$

$$\dot{r} = \frac{I_{13}}{I_{33}} \dot{p} + \frac{I_{NF}\ \dot{\Omega}_{NF}}{I_{33}} + \frac{\bar{y}}{I_{33}} (T_{F_1} \sin \theta_{L_1} - T_{F_2} \sin \theta_{L_2})$$

$$+ \frac{qSb}{I_{33}} C_{n_{\delta R}} \cdot \delta R \tag{4.201}$$

In these equations θ_L is the angle the fan thrust is rotated (θ_L is a function of louver angle) and $\bar{y}$ is the moment arm from the x-axis (parallel to the y-axis) to the center of either wing fan. θ_{L_1} and θ_{L_2} refer to louver angles at the port and starboard fans. Cross-coupling of angular rates is assumed negligible for the hover condition. The gyroscopic effects of the nose fan (NF) are included in equations (4.199), (4.200) and (4.201). These gyroscopic terms will be zero in conventional flight. Reaction moments are included in equations (4.199), (4.200) and (4.201) to account for aileron (δA), elevator (δE) and rudder (δR) control motion during hover. The reaction moments are contained in $C_{\ell_{\delta A}}$, $C_{m_{\delta E}}$ and $C_{n_{\delta R}}$ so that they phase naturally into the transition and conventional flight equations of motion. Pitch control is represented by nose fan thrust (T_{NF}). The pitching moment created by the nose fan is $f(T_{NF}) \cdot \ell_T$ where ℓ_T is the moment arm from the center of the nose fan to the c.g. of the aircraft.

Transitional Flight. As in the equations for transitional flight for the P.1127 (Section IV), the transitional equations for the XV-5A are composed of the equations (3.7) through (3.12). Consequently, we can write the equations of motion for the fan-in-wing directly.

EQUATIONS OF MOTION XV-5A. The equations of motion are expressed in aircraft body axes.

X Force Equation

$$m(\dot{U} + Wq_1 - Vr) = \frac{\rho V_B^2 S}{2}\left[C_x(\alpha, Ma) + C_x(\beta) + C_{x_{\delta F}} \cdot \delta F\right]$$

$$+ T_J \cos\alpha_R + T_F \sin\theta_L - mg\sin\theta$$

Y Force Equation

$$m(\dot{V} + Ur - Wp) = \frac{\rho V_B^2 S}{2}\left[C_{y_\beta} \cdot \beta + C_{y_{\delta R}} \cdot \delta R\right] + mg\cos\theta\sin\Phi$$

Z Force Equation

$$m(\dot{W} + Vp - Uq_1) = \frac{\rho V_B^2 S}{2}\left[C_z(\alpha, Ma) + C_{z_{\delta F}} \cdot \delta F + C_{z_{\delta E}} \cdot \delta E\right]$$

$$- T_J \sin\alpha_R - T_F\cos\theta_L + f(T_{NF}) + mg\cos\theta\cos\Phi$$

Ex
in

Roll Equation

$$\dot{p} = \frac{I_{13}}{I_{11}}(\dot{r} + pq_1) - \left(\frac{I_{33} - I_{22}}{I_{11}}\right) q_1 r + \frac{I_{NF}\,\Omega_{NF}}{I_{11}} q_1$$

$$+ \frac{\rho V_B^2 Sb}{2\, I_{11}}\left[C_{\ell_\beta} \cdot \beta + C_{\ell\delta A} \cdot \delta A + C_{\ell_{\delta R}} \cdot \delta R + \frac{b}{2V_B}\left(C_{\ell_p} \cdot p + C_{\ell_r} \cdot r\right)\right]$$

$$+ \frac{\bar{y}}{I_{11}}\left(T_{F_1}\cos\theta_{L_1} - T_{F_2}\cos\theta_{L_2}\right)$$

Pitch Equation

$$\dot{q}_1 = \frac{I_{13}}{I_{22}}(r^2 - p^2) - \left(\frac{I_{11} - I_{33}}{I_{22}}\right) pr - \frac{I_{NF}\ \Omega_{NF}}{I_{22}} p$$

$$+ \frac{\rho V_B^2 Sc}{2I_{22}} \left[C_m(\alpha, Ma) + C_{m_{\delta F}} \cdot \delta F + C_{m_{\delta E}} \cdot \delta E + \frac{c}{2V_B}\left(C_{m_{q_1}} \cdot q_1 + C_{m_{\dot{\alpha}}} \cdot \dot{\alpha}\right)\right]$$

$$+ \frac{f(T_{NF})\, \ell_T}{I_{22}}$$

Yaw Equation

$$\dot{r} = \frac{I_{13}}{I_{33}}(\dot{p} - q_1 r) - \left(\frac{I_{22} - I_{11}}{I_{33}}\right) pq_1 + \frac{I_{NF}\ \Omega_{NF}}{I_{33}}$$

$$+ \frac{\rho V_B^2 Sb}{2I_{33}} \left[C_{n_\beta} \cdot \beta + C_{n_{\delta A}} \cdot \delta A + C_{n_{\delta R}} \cdot \delta R + \frac{b}{2V_B}\left(C_{n_p} \cdot p + C_{n_r} \cdot r\right)\right]$$

$$+ \frac{\bar{y}}{I_{33}}\left(T_{F_1} \sin \theta_{L_1} - T_{F_2} \sin \theta_{L_2}\right)$$

External stores, rough air or landing gear conditions are not expressed in the above equations.

ROTATING THRUST

The Hawker P.1127 will be used as an example in the development of rotating thrust V/STOL simulation equations of motion. The equations of Section III as well as information from Hawker Aircraft Ltd. (Reference 9) will be directly applicable to the development of the simulation equations.

In order to obtain some idea of the P.1127 let us again consider Figure 5 and Table 8. In Figure 5, a three-view arrangement of the P.1127 is shown, whereas Table 8 contains some of the physical characteristics of the aircraft.

With this general idea of the P.1127, we may now develop the mathematical model for the P.1127. First, we define, if necessary, applicable axis systems. Second, we develop any additional aerodynamic coefficients. Finally, we write the equations of motion for the P.1127.

AXIS SYSTEMS FOR THE P.1127. The argument presented here is similar to that of the fan-in-wing. The only physical change in the configuration of the P.1127 from hover, through transition to normal flight is the rotation of the thrust nozzles. This rotation can be adequately described in the aircraft body axis system. In accord with Reference 9, body axes can be used during hover flight and stability axes can be used during transition and conventional flight in order to specify stability and control derivatives. Consequently, no axis systems in addition to the inertial, body, stability or wind axes will be necessary to develop equations of motion for the P.1127.

AERODYNAMIC FORCES AND MOMENTS - P.1127. In Reference 9, Hawker Aircraft lists equations of motion for hover and transitional flight. During hover, control is maintained by the reaction controls which obtain control power from the cockpit stick and rudder pedals. There are two roll reaction controls (one near each wing tip) which operate in conjunction with the ailerons from command signals generated by the cockpit stick. Next, there are two pitch reaction controls (one under the nose and one under the tail of the aircraft) which operate in conjunction with the all movable tail plane from command signals generated by the cockpit stick. Finally, there are two yaw reaction controls (one on each side of the tail of the aircraft) which operate in conjunction with the rudder from command signals generated by the rudder pedals.

Weight	
Basic	10,280 lbs.
Design Gross	15,749 lbs.
Wing	
Area	186.40 ft.^{2}
Span	22.82 ft.
Incidence to fuselage datum	1.75°
Dihedral Angle	-12°
Aerodynamic Mean Chord	8.667 ft.
Horizontal Tail (all movable)	
Area	44.20 ft.^{2}
Span	12.00 ft.
Dihedral Angle	-15°.50'
Fuselage	
Overall Length	42.00 ft.
Overall Height	10.75 ft.
Vertical Tail	
Area	26.10 ft.^{2}
Span	5.625
Height from fuselage datum	8.167 ft.
Underfin	
Area	6.0 ft.^{2}
Distance of tip from fuselage datum	1.67 ft.
Flaps	
Area	13.25 ft.^{2}
Movement	0 to 50°
Reaction Controls	
Distance of roll reaction nozzles from fuselage C_L centerline	11.20 ft.
Distance of forward pitch reaction nozzle from moment reference center	17.55
Distance of rear pitch reaction nozzle from moment reference center	20.55 ft.

Table 8. Selected Physical Characteristics P.1127

Reaction Controls (Cont'd.)

Distance of yaw reaction nozzles from moment reference center.	20.15 ft.

Aerodynamic Controls

Aileron Area (Port and Starboard)	8.75 $ft.^2$
Movement	$\pm$ 12°
Rudder Area	5.25 $ft.^2$
Movement	+ 15°
Tailplane Net Area	38.08 ft^2
Movement	$\pm$ 12°, - 10°
Trim Range	+ 5 1/2°, - 2 1/2°

Intake

Area	9.3 $ft.^2$

Powerplant

No. and Model	(1) Pegasus B.S. Pg. 5
Type	Ducted fan lift/thrust engine
Manufacturer	Bristol Siddeley
By-Pass Ratio	1.4
Length	8.25 ft.
Diameter	4.00 ft.
Maximum Engine Rating	18,000 lb.
Maximum Fan RPM (% design)	101.0

Table 8. Selected Physical Characteristics P.1127 (Cont'd.)

From Reference 9 further information concerning thrust is obtained. There are four jet nozzles operating from the engine. These nozzles are rotated simultaneously. The front port and starboard nozzles are cold jets while the aft port and starboard nozzles are hot jets.

The cold jets are oriented 5^{o} out from the x-z body axis plane. The hot jets are a function of θ_j and are given by the following expression:

$$\Phi_H = 7.5^{o} + 0.05\ \theta_j \tag{4.202}$$

The total thrust vector (four nozzles) is:

$$T = T_{GC} \cos 5^{o} + T_{GH} \cos \Phi_H$$

$$T = 0.996\ T_{GC} + T_{GH} \cos \Phi_H \tag{4.203}$$

where T_{GC} is the gross thrust from the two cold jet nozzles and T_{GH} is the gross thrust from the two hot jet nozzles. We will turn to aerodynamic forces and moments during hover and transitional flight.

Hover Flight. The equations used by Hawker in Reference 9 for hover are next stated. Nomenclature has been changed in these equations to agree with usage in this report.

$$\dot{p} = \frac{L_A}{I_{11}} \cdot A \tag{4.204}$$

$$\dot{q}_1 = \frac{M_E}{I_{22}} \cdot E \tag{4.205}$$

$$\dot{r} = \frac{N_R}{I_{33}} \cdot R \tag{4.206}$$

$$\dot{U} = \frac{g}{w} T \cos (\theta_j + \alpha_T) - g \sin \theta \tag{4.207}$$

$$\dot{V} - g \cos \theta \sin \Phi \tag{4.208}$$

$$\dot{W} = \frac{g}{w} T \sin (\theta_j + \alpha_T) + g \cos \theta \sin \Phi \tag{4.209}$$

In the above equations $\dot{p} = \frac{d^2 \Phi}{dt^2}$, $\dot{q}_1 = \frac{d^2 \theta}{dt^2}$ and $\dot{r} = \frac{d^2 \Psi}{dt^2}$. L_A, M_E, and N_R are aerodynamic moments due to reaction controls where A is the aileron, E is the tailplane and R the rudder. θ_j is the angle of deflection of the engines nozzles from the centerline ($\not{C}_L$) of the engine, a_T is the angle between $\not{C}_L$ and the x body axis and T is engine thrust.

Equations (4.204) through (4.209) will be accepted as hover equations and incorporated in the final equations of motion.

Transitional Flight. Again we turn to Reference 9 and quote here the transitional flight equations. These equations will be stated in the nomenclature used in this report. Appropriate comments will be made for each equation so as to compare it with the math model of Section III. There are no explicit thrust terms in the X and Z forces. The thrust terms will be in the final equations -- (4.216) through (4.221).

X - Force.

$$m\dot{U} = qS\,[C_{x_U} + C_{x_\alpha} + C_{x_{\delta F}} \cdot \delta F] - mg\,(\theta) \qquad (4.210)$$

Comparing equation (4.210) to (3.7), there is no $C_x(\beta)$ coefficient in (4.210). In (4.210), $C_{x_U} + C_{x_\alpha}$ corresponds to $C_x(\alpha, Ma)$ of (3.7) and mg sin $\theta \approx$ mg (θ) where θ is a small angle.

Y - Force.

$$m\dot{V} = qS\,[C_{y_\beta} \cdot \beta + C_{y_{\delta A}} \cdot \delta A + C_{y_{\delta R}} \cdot \delta R] + mg\,(\Phi) \qquad (4.211)$$

Comparing equation (4.211) to (3.8), there is no $C_{y_{\delta A}}$ of $C_{y_{\delta R}}$ in (3.8). In jet aircraft $C_{y_{\delta R}}$ and $C_{y_{\delta A}}$ are usually small. They are included in (4.211) because of their importance in the transition region. The term mg cos θ sin $\Phi \approx$ mg (Φ) where θ and Φ are considered small angles.

Z - Force.

$$m\dot{W} = qS\,[C_{z_U} + C_{z_\alpha} + C_{z_{\delta F}} \cdot \delta F] + mUq_1 \qquad (4.212)$$

Comparing (4.212) to (3.9), there is no $C_{z_{\delta E}} \cdot \delta E$ term in (4.218). This term should be in the final equations since δE does affect the Z - Force. The term, $C_{z_U} + C_{z_\alpha}$ corresponds to $C_z(\alpha, Ma)$.

In the moment expressions Hawker has assumed that cross-coupling of angular rates is negligible so the terms like $pq_1 \approx 0$ and $r^2 \approx 0$.

Roll.

$$\dot{p} - \frac{I_{13}}{I_{11}}\dot{r} = \rho \frac{V_B^2 Sb}{2I_{11}} [C_{\ell_\beta} \cdot \beta + C_{\ell_{\delta A}} \cdot \delta A + C_{\ell_{\delta R}} \cdot \delta R$$

$$+ \frac{b}{2V_B} C_{\ell_p} \cdot p + \frac{b}{2V_B} C_{\ell_r} \cdot r] \qquad (4.213)$$

Equation (4.213) is the same as equation (3.10) if cross-coupling of angular rates is neglected.

Pitch.

$$\dot{q}_1 = \rho \frac{V_B^2 Sc}{2I_{22}} [C_{m_U} + C_{m_\alpha} + C_{m_{\delta E}} \cdot \delta E + \frac{c}{2V_B} C_{m_{q_1}} \cdot q_1$$

$$+ \frac{c}{2V_B} C_{m_{\dot{\alpha}}} \cdot \dot{\alpha}] \qquad (4.214)$$

Yaw.

$$\dot{r} - \dot{p}\frac{I_{13}}{I_{33}} = \rho \frac{V_B^2 Sb}{2I_{33}} [C_{n_\beta} \cdot \beta + C_{n_{\delta A}} \cdot \delta A + C_{n_{\delta R}} \cdot \delta R$$

$$+ \frac{b}{2V_B} C_{n_p} \cdot p + \frac{b}{2V_B} C_{n_r} \cdot r] \qquad (4.215)$$

Equation (4.215) is the same as equation (3.12) if cross-coupling of angular rates is neglected.

From the hover equations and transitional equations final force and moment expressions will be written.

EQUATIONS OF MOTION - P.1127. The equations of motion herein presented are written by considering the equations (3.7) through (3.12) and the Hawker hovering and translational equations. In equations (3.7), (3.8) and (3.9), the velocity, U, is predominant. The equations are written in body axes.

X Force Equation.

$$m(\dot{U} + Wq_1 - Vr) = \rho \frac{V_B^2 S}{2} [C_{x_U} + C_{x_\alpha} + C_{x_{\delta F}} \cdot \delta F + C_x (\beta)]$$

$$+ T\cos(\theta_J + \alpha_T) - mg\sin\theta \qquad (4.216)$$

Y Force Equation

$$m\,(\dot{V} + Ur - Wp) = \rho \frac{V_B^2 S}{2} [C_{y_\beta} \cdot \beta + C_{y_{\delta A}} \cdot \delta A + C_{y_{\delta R}} \cdot \delta R]$$

$$+ mg \cos\theta \sin\Phi \qquad (4.217)$$

Z Force Equation.

$$m\,(\dot{W} + V_p - Uq_1) = \rho \frac{V_B^2 S}{2} [C_{z_U} + C_{z_\alpha} + C_{z_{\delta F}} \cdot \delta F + C_{z_{\delta E}} \cdot \delta E]$$

$$- T \sin(\theta_j + \alpha_T) + mg \cos\theta \cos\Phi \qquad (4.218)$$

Roll Equation.

$$\dot{p} = \frac{I_{13}}{I_{11}} (\dot{r} + pq_1) - \left(\frac{I_{33} - I_{22}}{I_{11}}\right) q_1 r + \frac{L_A}{I_{11}} + \rho \frac{V_B^2 Sb}{2I_{11}} [C_{\ell_\beta} \cdot \beta + C_{\ell_{\delta A}} \cdot \delta A$$

$$+ C_{\ell_{\delta R}} \cdot \delta R + \frac{b}{2V_B} C_{\ell_p} \cdot p + \frac{b}{2V_B} C_{\ell_r} \cdot r] \qquad (4.219)$$

Pitch Equation.

$$\dot{q}_1 = \frac{I_{13}}{I_{22}} (r^2 - p^2) - \left(\frac{I_{11} - I_{33}}{I_{22}}\right) pr + \frac{M_E}{I_{22}} \cdot E + \rho \frac{V_B^2 Sc}{2I_{22}} [C_{m_U} + C_{m_\alpha} + C_{m_{\delta E}} \cdot \delta E$$

$$+ C_{m_{\delta F}} \cdot \delta F + \frac{c}{2V_B} C_{m_{q_1}} \cdot q_1 + \frac{c}{2V_B} C_{m_{\dot{\alpha}}} \cdot \dot{\alpha}] \qquad (4.220)$$

Yaw Equation.

$$\dot{r} = \frac{I_{13}}{I_{33}} (\dot{p} - qr_1) - \left(\frac{I_{22} - I_{11}}{I_{33}}\right) p\, q_1 + \frac{N_R}{I_{33}} \cdot R + \frac{V_B^2 Sb}{2I_{33}} [C_{n_\beta} \cdot \beta$$

$$+ C_{n_{\delta A}} \cdot \delta A + C_{n_{\delta R}} \cdot \delta R + \frac{b}{2V_B} C_{n_p} \cdot p + \frac{b}{2V_B} C_{n_r} \cdot r] \qquad (4.221)$$

For $W \approx 0$, $V \approx 0$ and a small value of U, equations (4.216) through (4.221) are equivalent to the hover equations (4.204) through (4.209) since all the aerodynamic coefficients in brackets are practically zero.

As U becomes greater (θ_j approaches zero) $L_{\underline{A}}$, M_E, and N_R phase out and the aerodynamic terms are predominant as in equations (4.210) through (4.215). The transitional equations then carry directly into conventional flight. External stores, rough air, or landing gear conditions are not developed in equations (4.216) through (4.221).

SECTION V

BIBLIOGRAPHY

1. Breul, Harry T., A Simulator Study of Tilt-Wing Handling Qualities, Preliminary Report, Grumman Research Department.

2. Connelly, Mark E., Simulation of Aircraft, Report 7591-R-1, 15 Feb. 1958.

3. Constant, F.W., Theoretical Physics, Addison-Wesley Pub. Co., Cambridge, Mass. 1954.

4. Curtiss-Wright Corporation, Curtiss-Wright Model 200 High-Speed VTOL, Brochure, May 1962.

5. Dommasch D.O., Sherby S.S., and Connolly T.F., Airplane Aerodynamics, Second Edition, 1957.

6. Ling-Temco-Vought, Unbound notes, XC-142 Flying Qualities Simulator Setup, Dir. No: 2-53310/210-42 (S & C-18).

7. Northrop Aircraft, Inc., Dynamics of the Airframe, BuAer Report AE-61-4II, September 1952.

8. Soule, W. L. and Baumgarten, M.F., A Method for Estimating the Aerodynamic Forces and Moments Acting at the Center of Gravity of a Tilt-Duct Type VTOL Aircraft, Grumman RM-211, August 1962.

9. P.O.N. No. 614, Hawker Aircraft LTD.

APPENDIX A

DERIVATION OF BASIC EQUATIONS OF MOTION

The derivation of the equations of motion for an aircraft flying in a conventional manner has been done by means of a notation that is economical of space. Naturally, there is a sophisticated degree of abstractness in manipulating equations developed in this notation; however, no more than the simplest best known expressions in mechanics and the representation of direction cosines are actually being used. The idea being developed in an equation can be perceived as a whole using this notation instead of seeing bits and pieces of derivation. The main terms of an equation are immediately apparent, and initial assumptions as to importance of various terms in the equation can be applied in order to give an orderly routine of simplification of the general equation for a particular ~~set of~~ assumptions.

NOTATION

The notation employed uses the device of subscripting and superscripting a particular symbol with an algebraic variable which will vary over a prescribed range. It is actually the range and summation techniques developed in Tensor Analysis. More precisely, what is used and manipulated are Cartesian Tensors. These are particularly simple tensors in three dimensions using Cartesian coordinates. In fact, for the purposes of these derivations the only tensor quality is the adaption of the range and summation conventions plus a few manipulative methods, and the advantage gained from Tensor Analysis is a concise notation.

e - SYMBOL

e_{ijk} or e_{abc} is defined in the following manner:

e_{ijk} = +1 if ijk is an even permutation of 123

e_{ijk} = -1 if ijk is an odd permutation of 123

e_{ijk} = 0 if i = j or j = k or i = k or i = j = k

For example:

$$+1 = e_{123} = e_{231} = e_{312}$$

$$-1 = e_{132} = e_{321} = e_{213}$$

$$0 = e_{112} = e_{122} = e_{121} = e_{111}$$

KRONECKER DELTAS (δ)

First Order Kronecker Deltas

$$\delta^i_j = +1 \text{ if } i = j$$

$$\delta^i_j = 0 \text{ otherwise } (i \neq j)$$

Second Order Kronecker Deltas

$$\delta^{ab}_{ij} = +1 \text{ if } a \neq b,\ i \neq j,\ a = i,\ b = j$$

$$\delta^{ab}_{ij} = -1 \text{ if } a \neq b,\ i \neq j,\ a = j,\ b = i$$

$$\delta^{ab}_{ij} = 0 \text{ otherwise}$$

This scheme can be generalized to n^{th} order Kronecker Deltas. However, since the coordinate systems being used in this report are three dimensional, the highest order Kronecker Delta encountered will be three.

Third Order Kronecker Deltas

$\delta^{abc}_{ijk} = +1$ if $a \neq b \neq c$, $a \neq c$ and ijk is an even permutation of abc.

$\delta^{abc}_{ijk} = -1$ if $a \neq b \neq c$, $a \neq c$ and ijk is an odd permutation of abc.

$\delta^{abc}_{ijk} = 0$, otherwise

Product of Two e Symbols

$$e_{abc}e_{ijk} = \delta^{ijk}_{abc}$$

In particular a special case is

$$e_{ajc}e_{ijk} = \delta^{ajc}_{ijk} = \delta^{ac}_{ik}$$

Example:

Does $e_{123}e_{321} = \delta^{13}_{31}$?

$$(+1)\ (-1) = -1$$

$$-1 = -1 \qquad \text{Q.E.D.}$$

Expansion of Second Order Kronecker Delta

$$\delta^{ab}_{ij} = \delta^{a}_{i}\delta^{b}_{j} - \delta^{a}_{j}\delta^{b}_{i}$$

Example:

Does $\delta^{12}_{21} = \delta^{1}_{2}\delta^{2}_{1} - \delta^{1}_{1}\delta^{2}_{2}$?

$$(-1) = (0)\ (0) - (1)\ (1)$$

$$(-1) = (-1) \qquad \text{Q.E.D.}$$

Example:

$$\delta^{11}_{23} = \delta^{1}_{2}\delta^{1}_{3} - \delta^{1}_{3}\delta^{1}_{2}$$

$$0 = (0)\ (0) - (0)\ (0)$$

$$0 = 0$$

e - Symbol Times Kronecker Delta

$$\delta^{a}_{d}e_{jmd} = e_{jma}$$

Example:

$$\delta^{1}_{1}e_{jm1} = e_{jm1}$$

But $\delta^{1}_{1} = +1$

$$e_{jm1} = e_{jm1}$$

Example:

$$\delta^{1}_{2}e_{jm2} = e_{jm1} \qquad \delta^{1}_{2} = 0$$

Possibilities

$$\delta^{1}_{2}e_{312} = e_{311}$$

$$0 = 0$$

or

$$\delta^{1}_{2}e_{132} = e_{131}$$

RANGE AND SUMMATION CONVENTIONS. The definitions are as follows:

Range Convention. When an uncapitalized letter (superscript or subscript) occurs unrepeated in a term, it is understood to take all the values 1, 2 and 3, where 3 is the number of dimensions of the space; i.e., three dimensional space.

Summation Convention. When an uncapitalized letter (superscript or subscript) occurs repeated in a term, then summation with respect to this letter is understood, where the extent of summation is 1...3.

For example, consider the following expression:

$$F^{xj} = M(\dot{V}^{xj} + V^{xk}\omega^{xc}e_{jck})$$

In F^{xj}, the j indicates a range: j = 1, 2 or 3

So there is:

$$F^{X1} = M(\dot{V}^{X1} + V^{xk}\omega^{xc}e_{1ck})$$

$$F^{X2} = M(\dot{V}^{X2} + V^{xk}\omega^{xc}e_{2ck})$$

$$F^{X3} = M(\dot{V}^{X2} + V^{xk}\omega^{xc}e_{3ck})$$

Notice now $V^{xk}\omega^{xc}e_{jck}$. In F^{X1}, F^{X2} and F^{X3} there are two repeated indices--c and k. This would indicate a summation process according to the summation convention:

$$F^{X1} = \sum_{c,k=1}^{3} M\,(\dot{V}^{X1} + V^{xk}\omega^{xc}e_{1ck})$$

$$F^{X2} = \sum_{c,k=1}^{3} M\,(\dot{V}^{X2} + V^{xk}\omega^{xc}e_{2ck})$$

$$F^{X3} = \sum_{c,\,k=1}^{3} M\,(\dot{V}^{X3} + V^{xk}\omega^{xc}e_{3ck})$$

Expanding F^{X1}, it is written:

$$\begin{aligned} F^{X1} = (\dot{V}^{X1} &+ V^{X1}\omega^{X1}e_{111} + V^{X2}\omega^{X1}e_{112} + V^{X3}\omega^{X1}e_{113} \\ &+ V^{X1}\omega^{X2}e_{121} + V^{X2}\omega^{X2}e_{122} + V^{X3}\omega^{X2}e_{123} \\ &+ V^{X1}\omega^{X3}e_{131} + V^{X2}\omega^{X3}e_{132} + V^{X3}\omega^{X3}e_{133})M \end{aligned}$$

Collect non-vanishing terms so that

$$F^{X1} = (\dot{V}^{X1} + V^{X3}\omega^{X2}e_{123} + V^{X2}\omega^{X3}e_{132})M$$

$$F^{X1} = (\dot{V}^{X1} + V^{X3}\omega^{X2} - V^{X2}\omega^{X3})M$$

F^{X2} and F^{X3} are treated in a similar manner.

In the above expansion of F^{X1}, the summation and range conventions were used. It should be noted that there is an easier method to arrive at the expanded F^{X1}.

$$F^{X1} = M(\dot{V}^{X1} + V^{xk}\omega^{xc}e_{1ck})$$

Note: e_{1ck} to be non zero can only have c and k equal to 2 or 3. That is:

$$e_{1ck} = e_{123} \text{ and } e_{132}$$

Then for F^{X1} we write the non-vanishing terms immediately:

$$F^{X1} = M(\dot{V}^{X1} + V^{X3}\omega^{X2}e_{123} + V^{X2}\omega^{X3}e_{132})$$

$$F^{X1} = (\dot{V}^{X1} + V^{X3}\omega^{X2} - V^{X2}\omega^{X3})M$$

Thus, by noting the non zero properties of the e - symbol, the summation process can be shortened.

CONTENTS OF NOMENCLATURE. This is a description of the more important individual designations that will appear in the terms of the equations.

Position - Particle to Particle. (1) This is designated by a vector $\vec{r}$. In the indexed notation $\vec{r}$ becomes r^{ti}, which indicates that $\vec{r}$ originates at the origin of the t-axes. There may be other vectors such as r^{si} or r^{ni}.

(2) $i = 1, 2, 3$ gives the particular component of r^{ti}.

(3) To differentiate between various r^{ti}, a subscript is added. There results r_o^{ti}, r_1^{ti}, r_2^{ti}, ... or for a particle: r_p^{ti}.

Velocity. This is:

$$V^{ti} = \dot{r}_o^{ti}$$

$$V_1^{ti} = \dot{r}_1^{ti}$$

$$V_2^{ti} = \dot{r}_2^{ti}$$

and for a particle $V_p^{ti} = \dot{r}_p^{ti}$

All dots over symbols refer to derivatives with respect to time.

Remember that ti can be replaced by yi, zi, si or some other set of coordinates.

Acceleration. This is:

$$a_o^{si} = \dot{V}_1^{si} = \ddot{r}_o^{si}$$

$$a_1^{si} = \dot{V}_1^{si} = \ddot{r}_1^{si}$$

and for a particle $a_p^{si} = \dot{V}_p^{si} = \ddot{r}_p^{si}$

Mass. A mass point is designated as m_p. When deriving equations of motion a summation of mass points is made.

Example:

For fuselage particle: $\int m_p = M_F$(mass of fuselage)

Force. In general: $\vec{F} = m\,\vec{a}$ (vector form)

Now in indexed notation for example: $f_p^{ti} = m_p a_p^{ti}$

But the equations are developed by knowing what particles are being manipulated. If the expression for the fuselage particle is being sought the final form will be:

$f_p^{ti} = m_p a_p^{ti}$, where a_p^{ti} is acceleration and is a function of direction cosines, position vectors, and rates of change of both.
Integrating: $\int f_p = F_F$ Subscript F fuselage

$$\int m_p = M_F \quad M_F \equiv \text{Mass of fuselage particle}$$

Torque. Vectorial: $\vec{T} = \vec{r} \times \vec{F}$

Now in indexed notation: $t^{sm} = m_p a_p^{sj} r_p^{sd} e_{jmd} = f_p^{sj} r_p^{sd}$

Integrating: $\int t^{sm} = T_F^{sm}$ if a_p^{sj} is the acceleration associated with the fuselage particle.

Inertia. Vectorial: The angular mementum $\vec{J} = \sum_{i=1}^{n} [m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)]$

$$\text{or } \vec{J} = \vec{i}[\omega_x \sum_{i=1}^{n} m_i(y_i^2 + z_i^2) - \omega_y \sum_{i=1}^{n} m_i x_i y_i - \omega_z \sum_{i=1}^{n} m_i x_i z_i]$$

$$+ \vec{j}[-\omega_x \sum_{i=1}^{n} m_i x_i y_i + \omega_y \sum_{i=1}^{n} m_i(x_i^2 + z_i^2) - \omega_z \sum_{i=1}^{n} m_i y_i z_i]$$

$$+ \vec{k}[-\omega_x \sum_{i=1}^{n} m_i x_i z_i - \omega_y \sum_{i=1}^{n} m_i y_i z_i + \omega_z \sum_{i=1}^{n} m_i(x_i^2 + y_i^2)]$$

Moments of inertia are terms like:

$$\sum m_i(y_i^2 + z_i^2) = I_{xx}$$

Products of inertia are terms like:

$$\sum m_i x_i y_i = I_{xy}$$

Now in indexed notation: $I_{pq} = \int m_p r_p^{xa} r_p^{xd} e_{pak} e_{qdk}$; p, q fixed

$$\text{or} \quad \int m_p r_p^{xa} r_p^{xd} = -I_{ad} + \frac{1}{2}(I_{11} + I_{22} + I_{33})\delta_d^a$$

I_{pq} defines both moments and products of inertia. To make this clear consider the following examples:

$$I_{pq} = \int m_p r_p^{xa} r_p^{xd} e_{pak} e_{qdk} = \int m_p r_p^{xa} r_p^{xd} \delta_{pa}^{qd}$$

Example: Moment of inertia

$$I_{11} = \int m_p r_p^{xa} r_p^{xd} \delta_{1a}^{1d}$$

Now δ_{1a}^{1d}, δ_{12}^{12}, and δ_{13}^{13} are non zero

Then:

$$I_{11} = \int m_p r_p^{X2} r_p^{X2} \delta_{12}^{12} + \int m_p r_p^{X3} r_p^{X3} \delta_{13}^{13}$$

$$\delta_{12}^{12} = +1, \ \delta_{13}^{13} = +1$$

$$I_{11} = \int m_p [(r_p^{X2})^2 + (r_p^{X3})^2]$$

Example: Product of inertia

$$I_{12} = \int m_p r_p^{xa} r_p^{xd} \delta_{1a}^{2d}$$

Now $\delta_{1a}^{2d} = \delta_{12}^{21} = -1$

Then:

$$I_{12} = -\int m_p r_p^{X2} r_p^{X1}$$

Example:

[illegible]

Let a = 1 and d = 1

$$\int m_p r_p^{X1} r_p^{X1} = -I_{11} + \frac{1}{2}(I_{11} + I_{22} + I_{33})\delta_1^1$$

$$\delta_1^1 = +1$$

$$I_{11} = \int m_p[(r_p^{X2})^2 + (r_p^{X3})2]$$

$$I_{22} = \int m_p[(r_p^{X3})^2 + (r_p^{X1})^2]$$

$$I_{33} = \int m_p[(r_p^{X1})^2 + (r_p^{X2})^2]$$

So that:

$$\int m_p r_p^{X1} r_p^{X1} = -\frac{I_{11}}{2} + \frac{I_{22}}{2} + \frac{I_{33}}{2}$$

$$\int m_p r_p^{X1} r_p^{X1} = -\frac{\int m_p r_p^{X2} r_p^{X2}}{2} - \frac{\int m_p r_p^{X3} r_p^{X3}}{2} + \frac{\int m_p r_p^{X3} r_p^{X3}}{2} + \frac{\int m_p r_p^{X1} r_p^{X1}}{2} + \frac{\int m_p r_p^{X1} r_p^{X1}}{2} + \frac{\int m_p r_p^{X2} r_p^{X2}}{2}$$

AXES SYSTEMS

Two different axis systems are described--the inertial system and the fuselage system in terms of direction cosines. Direction cosines are the angles which define a vector in a particular axes system. The matrices developed in this appendix relate the measurement of a unit length in one direction as measured in a different direction. This is accomplished by the reorientation of one axis system with respect to another axes system. The method used is to consider that the axes systems have common origins and then by rotating the axes of one system with respect to the other achieve coincidence. The direction cosines measure this rotation.

In this report objects in three dimensional space are described by the use of Cartesian coordinates which are orthogonal by definition. Orthogonality allows the transpose of the direction cosine matrix to equal its inverse. It is sufficient to describe the rotation of one axes system with respect to another by the use of three angles (Euler angles).

INERTIAL AXES SYSTEM. The system is identified by $N\alpha$ axes (α = 1, 2, 3).

N1 is in the horizontal plane with positive direction North.

N2 is in the horizontal plane with positive direction East.

N3 is normal to the N1N2 plane with positive direction down.

THE FUSELAGE AXES SYSTEM. The system is identified by $x\alpha$ axes (α = 1,2,3). This axes system corresponds to the body axes system used in normal aircraft parlance. The origin is fixed in the fuselage (usually at the nominal center of mass). The origin of the x-axes is separated from the origin of the N-axes by a vector $\vec{r}_o$--see Figure A1.

x1 is in the horizontal plane with positive direction noseward.

x1 is not necessarily parallel to the water line axis defined by the aircraft manufacturer.

x2 is in the horizontal plane with positive direction right.

x3 is normal to the x1x2 plane with positive direction down.

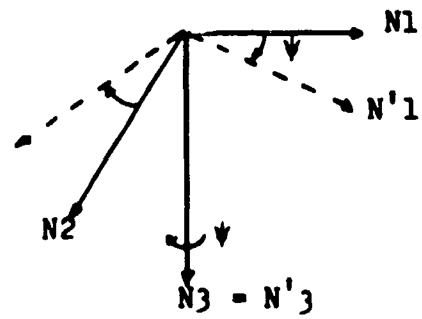
INERTIAL TO FUSELAGE AXES DIRECTION COSINE MATRICES. Three angles, the Euler angles are sufficient to locate the fuselage in inertial space. The inertial system is translated such that its origin coincides with the origin of the fuselage (i.e. the nominal c.m. of the aircraft). Specific rotations of the N-axes are then made so that a unit of measure in the N-axes can be expressed in the x-axes. In each case the particular rotation is in a plane so that there is a rotation about the third axis--that is the rotation written as a matrix is simply sines and cosines of the rotating angle with the orthogonal property of the axes systems stating that $\sin^2 + \cos^2 = 1$. This will become clear.

The three rotations are ψ, θ and Φ.

Yaw by Angle ψ Around N3 Axis. Here an intermediate system N' is defined by the rotation of ψ around N3.

Define C as direction cosine matrix.

$$C^{N\alpha}_{N'i} = \begin{array}{c|ccc} & N'1 & N'2 & N'3 \\ \hline N1 & \cos\psi & -\sin\psi & 0 \\ N2 & \sin\psi & \cos\psi & 0 \\ N3 & 0 & 0 & 1 \end{array}$$



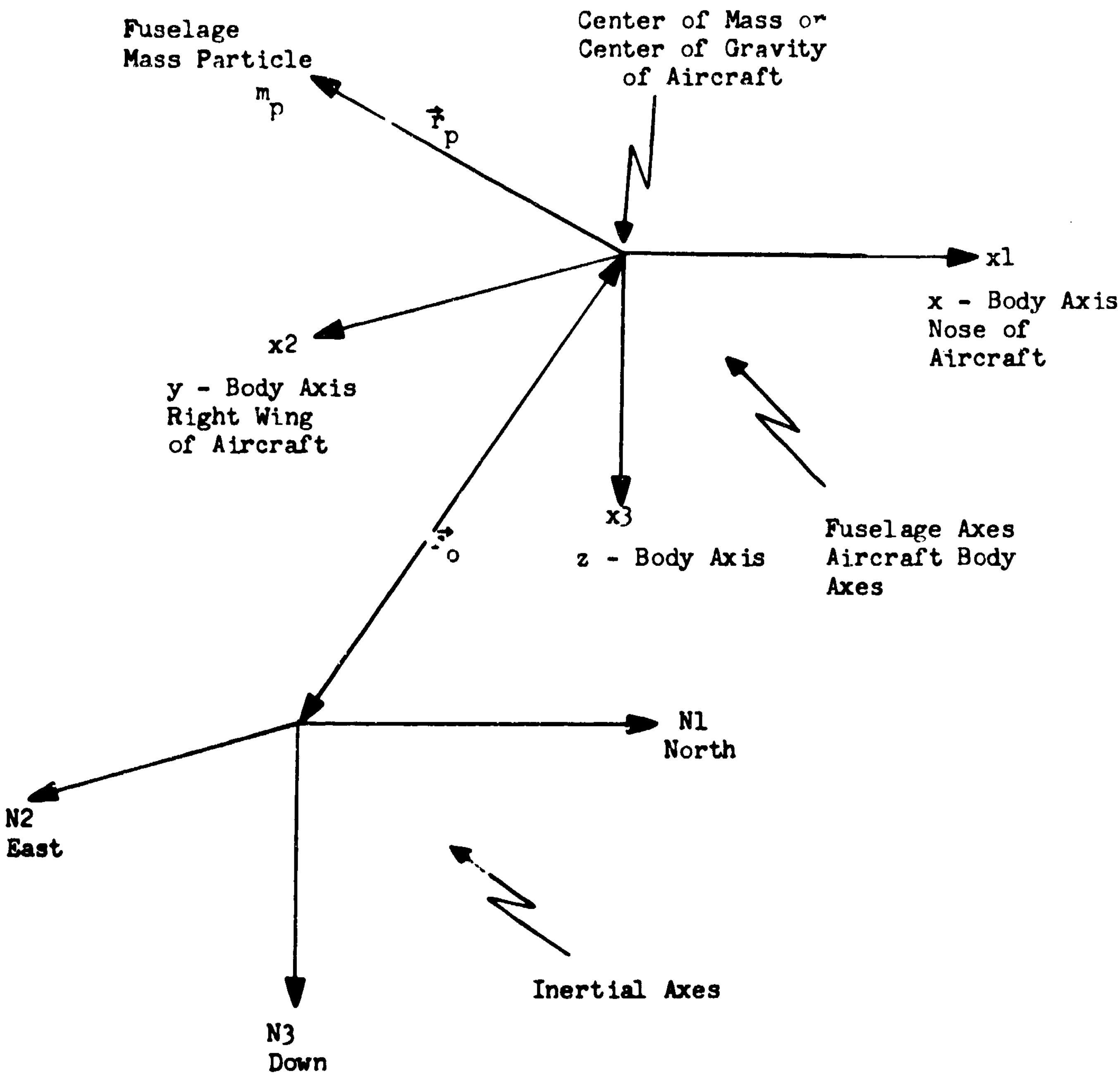
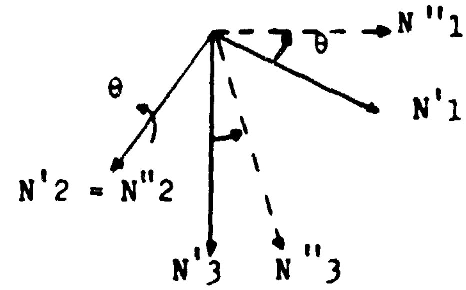


Figure A1. Inertial and Fuselage Axes

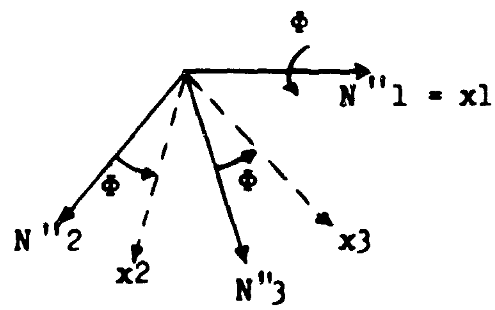
Pitch by Angle θ Around N'2 Axis. Here an intermediate system N'' is defined by the rotation of θ around N'2.

$$C_{N''\xi}^{N'\gamma} = \begin{array}{c} \\ N'1 \\ N'2 \\ N'3 \end{array} \begin{array}{ccc} N''1 & N''2 & N''3 \\ \hline \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{array}$$



Roll by Angle Φ Around N''1 Axis. This third rotation arrives at the x-axes.

$$C_{x\beta}^{N''\xi} = \begin{array}{c} \\ N''1 \\ N''2 \\ N''3 \end{array} \begin{array}{ccc} x1 & x2 & x3 \\ \hline 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{array}$$



Combination of Three Rotations (ψ, θ, Φ). To get the final result--the relation between the inertial coordinate system and the fuselage coordinate system--a matrix multiplication is performed.

This multiplication is

$$C_{N'\gamma}^{N\alpha} \times C_{N''\xi}^{N'\gamma} \times C_{x\beta}^{N''\xi} = C_{x\beta}^{N\alpha}$$

$$\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{bmatrix}$$

$$\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & \sin\theta\ \sin\theta & \sin\theta\ \cos\Phi \\ 0 & \cos\Phi & -\sin\Phi \\ -\sin\theta & \cos\theta\ \sin\Phi & \cos\theta\ \cos\Phi \end{bmatrix} = C_{x\beta}^{N\alpha}$$

	x1	x2	x3
N1	$\cos\psi \cos\theta$	$\cos\psi \sin\theta \sin\Phi - \sin\psi \cos\Phi$	$\cos\psi \sin\theta \cos\Phi + \sin\psi \sin\theta$
N2	$\sin\psi \cos\theta$	$\sin\psi \sin\theta \sin\Phi + \cos\psi \cos\Phi$	$\sin\psi \sin\theta \cos\Phi - \cos\psi \sin\Phi$
N3	$-\sin\theta$	$\cos\theta \sin\Phi$	$\cos\theta \cos\Phi$

$= C_{x\beta}^{N\alpha}$

Note that $C_{x\beta}^{N\alpha} C_{N\alpha}^{x\beta} = 1$

where $C_{N\alpha}^{x\beta} = (C_{x\beta}^{N\alpha})^{-1} = (C_{x\beta}^{N\alpha})^{T}$

$(\quad)^{-1}$ Inverse matrix

$(\quad)^{T}$ Transpose matrix

INERTIAL TO FUSELAGE AXES RATES. $\dot{\Phi}$, $\dot{\theta}$ and $\dot{\psi}$ are measured as p, q_1 and r around the x1, x2 and x3 axes respectively. The dot is the total derivative with respect to time.

$\dot{\psi}$ Occurs Around the N3 Axis. To get the direction cosine components refer to $C_{x\beta}^{N\alpha}$.

$p = \dot{\psi}(-\sin\theta)$

$q_1 = \dot{\psi} \cos\theta \sin\Phi$

$r = \dot{\psi} \cos\theta \cos\Phi$

$\dot{\theta}$ Occurs around the N'2 Axis. To get the direction cosine components refer to

$C_{N''\xi}^{N'\gamma} \times C_{x\beta}^{N''\xi} = C_{x\beta}^{N'\gamma}$

$p = \dot{\theta}\ (0)$

$q_1 = \dot{\theta} \cos\Phi$

$r = \dot{\theta}\ (-\sin\Phi)$

$\dot{\Phi}$ Occurs Around the N_1'' axis. To get the direction cosine components refer to $C_{x\beta}^{N''\xi}$.

$p = \dot{\Phi}\ (1)$

$q_1 = \dot{\Phi}\ (0)$

$r = \dot{\Phi}\ (0)$

Combination of Three Rates (p, q_1, r).

$$p = \dot{\psi}\,(-\sin\theta) \quad + \dot{\theta}\,(0) \quad + \dot{\Phi}\,(1)$$

$$q_1 = \dot{\psi}\cos\theta\sin\Phi \quad + \dot{\theta}\cos\Phi \quad + \dot{\Phi}\,(0)$$

$$r = \dot{\psi}\cos\theta\cos\Phi \quad + \dot{\theta}\,(-\sin\Phi) \quad + \dot{\Phi}\,(0)$$

In matrix notation this becomes

$$\begin{bmatrix} p \\ q_1 \\ r \end{bmatrix} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \cos\theta\sin\Phi & \cos\Phi & 0 \\ \cos\theta\cos\Phi & -\sin\Phi & 0 \end{bmatrix} \times \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\Phi} \end{bmatrix}$$

By forming the inverse there is

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\Phi} \end{bmatrix} = \begin{bmatrix} p \\ q_1 \\ r \end{bmatrix} \times \begin{bmatrix} 0 & \dfrac{\sin\Phi}{\cos\theta} & \dfrac{\cos\Phi}{\cos\theta} \\ 0 & \cos\Phi & -\sin\Phi \\ 1 & \tan\theta\sin\Phi & \tan\theta\cos\Phi \end{bmatrix}$$

DERIVATION OF FORCE AND TORQUE EQUATIONS

The derivation of force and torque equations is done in concise notation. Certain assumptions are present in the use of these aircraft equations of motion. These assumptions are as follows:

(a) The earth is assumed to be a flat plane with gravity acting normal to this plane and it is not rotating or translating through space. In the derivation of the equations of motion, Newton's law, $F = ma$, is used in its most simple form--there is no mass variation with change in time.

(b) The x1 and x3 fuselage axes define a plane of symmetry for the aircraft.

(c) There is a fixed center of gravity.

(d) There is no relative motion between the earth and its atmosphere.

The equations will now be developed using the inertial and fuselage axes, Newton's law, and direction cosines of the Eulerian angles.

FORCE ASSOCIATED WITH THE FUSELAGE PARTICLE. The position of m_p for the fuselage particle with respect to Ni frame of reference is stated as:

$$r^{Ni} = r_o^{Ni} + r_p^{xa} C_{xa}^{Ni} .$$ See Figure A1.

The inertial system velocity is given as:

$$V_p^{Ni} = \dot{r}^{Ni} = \dot{r}_o^{Ni} + \dot{r}_p^{xa} C_{xa}^{Ni} + r_p^{xa} \dot{C}_{xa}^{Ni}$$ by differentiation of r^{Ni}.

$$V_p^{Ni} = \dot{r}_o^{Ni} + r_p^{xa} \dot{C}_{xa}^{Ni}, \dot{r}_p^{xa} = 0$$ since r_p^{Ni} = constant

but $\dot{C}_{xa}^{Ni} = C_{xb}^{Ni} \omega^{xc} e_{abc}$, therefore

$$V_p^{Ni} = \dot{r}_o^{Ni} + r_p^{xa} [C_{xb}^{Ni} \omega^{xc} e_{abc}] \text{ by substitution} \qquad (A1)$$

Note: For vector treatment of this, refer to Reference 3, Theoretical Physics - Mechanics by F. W. Constant, pages 69 to 71.

The inertial system acceleration is given by a similar procedure as

$$a_p^{Ni} = \dot{V}_p^{Ni}$$

$$a_p^{Ni} = \ddot{r}_o^{Ni} + r_p^{xa} [C_{xb}^{Ni} \dot{\omega}^{xc} e_{abc} + \dot{C}_{xb}^{Ni} \omega^{xc} e_{abc}]$$

$$a_p^{Ni} = \ddot{r}_o^{Ni} + r_p^{xa} [C_{xb}^{Ni} \dot{\omega}^{xc} e_{abc} + C_{xh}^{Ni} \omega^{xg} e_{bhg} \omega^{xc} e_{abc}]$$

$$a_p^{Ni} = \ddot{r}_o^{Ni} + r_p^{xa} C_{xb}^{Ni} \dot{\omega}^{xc} e_{abc} + r_p^{xa} C_{xh}^{Ni} \omega^{xg} \omega^{xc} e_{bhg} e_{abc} . \qquad (A2)$$

For a particular mass particle m_p it can be said that

$$f_p^{Ni} = m_p a_p^{Ni} \qquad (A3)$$

and transforming to the x-axis

$$f_p^{xj} = m_p a_p^{Ni} C_{Ni}^{xj} . \qquad (A3)$$

Substituting (A2) into (A3) yields

$$f_p^{xi} = m_p [\ddot{r}_o^{Ni} + r_p^{xa} C_{xb}^{Ni} \dot{\omega}^{xc} e_{abc} + r_p^{xa} C_{xh}^{Ni} \omega^{xg} \omega^{xc} e_{bhg} e_{abc}] C_{Ni}^{xj}$$

Interchange b and h where they both appear in the same term and then by changing b to j,

$$f_p^{xj} = m_p [\ddot{r}_o^{Ni} C_{Ni}^{xj} + r_p^{xa} \dot{\omega}^{xc} e_{ajc} + r_p^{xa} \omega^{xg} \omega^{xc} e_{hjg} e_{ahc}] . \qquad (A4)$$

Now is written

$$V^{Ni} = \dot{r}_o^{Ni}$$

$$V^{xj} = \dot{r}_o^{Ni} C_{Ni}^{xj}$$

$$\dot{V}^{xj} = \ddot{r}_o^{Ni} C_{Ni}^{xj} + \dot{r}_o^{Ni} C_{Ni}^{xk} \omega^{xc} e_{jkc}, \qquad (A5)$$

$$V^{xk} = \dot{r}_o^{Ni} C_{Ni}^{xk} . \qquad (A6)$$

Substituting (A6) into (A5) gives

$$\dot{V}^{xj} = \ddot{r}_o^{Ni} C_{Ni}^{xj} + V^{xk} \omega^{xc} e_{jkc} .$$

Rearranging and letting $e_{jkc} = -e_{jck}$,

$$\ddot{r}_o^{Ni} C_{Ni}^{xj} = \dot{V}^{xj} + V^{xk} \omega^{xc} e_{jck} . \qquad (A7)$$

Substituting (A7) into (A4) offers

$$f_p^{xj} = m_p[(\dot{V}^{xj} + V^{xk} \omega^{xb} e_{jbk}) + r_p^{xa} \dot{\omega}^{xc} e_{ajc} + r_p^{xa} \omega^{xg} \omega^{xc} e_{ahc} e_{hjg}].$$

Integrating over f_p^{xj}, m_p and $m_p r_p^{xa}$ finally yields.

$$F^{xj} = M_F[\dot{V}^{xj} + V^{xk} \omega^{xc} e_{jck} + r_F^{xa} \dot{\omega}^{xc} e_{ajc} + r_F^{xa} \omega^{xg} \omega^{xc} e_{abc} e_{bjg}].$$

Since

$$e_{ahc} e_{hjg} = \delta_{ahc}^{hjg} = \delta_{hca}^{hjg} = \delta_{ca}^{jg} ,$$

$$F_F^{xj} = M_F(\dot{V}^{xj} + V^{xk} \omega^{xc} e_{jck}) + M_F r_F^{xa} (\dot{\omega}^{xc} e_{ajc} + \omega^{xc} \omega^{xg} \delta_{ca}^{jg}) . \qquad (A8)$$

TORQUE DUE TO THE FUSELAGE PARTICLE. The definition of torque offers:

$$t^{xm} \equiv m_p a_p^{xj} r_p^{xd} e_{jmd} = f_p^{xj} r_p^{xd} e_{jmd}.$$

That is, by knowing the force, the torque of a particle can be developed by substituting for f_p^{xj} the terms $m_p a_p^{xj}$ from (A8), realizing, of course, that this a_p^{xj} is the acceleration expression developed for the fuselage particle. Then it follows from the substitution that

$$t^{xm} = m_p r_p^{xd} [(\dot{V}^{xj} + V^{xk} \omega^{xc} e_{jck}) + r_p^{xa} \dot{\omega}^{xc} e_{ajc} + r_p^{xa} \omega^{xc} \omega^{xg} \delta_{ca}^{jg}] e_{jmd} \qquad (A9)$$

By substituting (A9) into the expression

$$\int t^{xm} = T_F^{xm} \tag{A10}$$

There is obtained

$$\begin{aligned} T_F^{xm} &= \int m_p r_p^{xd} (\dot{V}^{xj} + V^{xk}\omega^{xc} e_{jck}) e_{jmd} \\ &+ \int m_p r_p^{xa} r_p^{xd} \dot{\omega}^{xc} e_{ajc} e_{jmd} \\ &+ \int m_p r_p^{xa} r_p^{xd} \omega^{xc} \omega^{xg} \delta_{ca}^{jg} e_{jmd} \end{aligned} \tag{A11}$$

Consider the second and third terms of the expression (A11). These will be developed using the definition of inertia. The definition is:

$$I_{pq} = \int m_p r_p^{xa} r_p^{xd} e_{pak} e_{qdk}, \text{ with p, q fixed} \tag{A12}$$

$$I_{pq} = \int m_p r_p^{xa} r_p^{xd} \delta_{qd}^{pa} , \tag{A13}$$

or,

$$\int m_p r_p^{xa} r_p^{xd} = -I_{ad} + \frac{1}{2}(I_{11} + I_{22} + I_{33}) \delta_d^a . \tag{A14}$$

$$\int m_p r_p^{xa} r_p^{xd} \dot{\omega}^{xc} e_{ajc} e_{jmd} = I_{F}^{cm} \dot{\omega}^{xc} . \tag{A15}$$

Now is written

$$\begin{aligned} &\int m_p r_p^{xa} r_p^{xd} \omega^{xc} \omega^{xg} \delta_{ca}^{jg} e_{jmd} \\ &= (-I_{ad} + \frac{1}{2}(I_{11} + I_{22} + I_{33}) \delta_d^a)\ \omega^{xc} \omega^{xg} \delta_{ca}^{jg} e_{jmd} \end{aligned} \tag{A16}$$

and knowing that

$$\begin{aligned} \delta_d^a e_{jmd} &= e_{jma} \\ \delta_{ca}^{jg} &= \delta_c^j \delta_a^g - \delta_a^j \delta_c^g \end{aligned} \tag{A17}$$

Substitution of (A17) into (A16) gives

$$\begin{aligned} &\int m_p r_p^{xa} r_p^{xd} \omega^{xc} \omega^{xg} \delta_{ca}^{jg} e_{jmd} \\ &= -\omega^{xc} \omega^{xg} I_{ad} e_{jmd} (\delta_c^j \delta_a^g - \delta_a^j \delta_c^g) \\ &+ \frac{1}{2}(I_{11} + I_{22} + I_{33})\ (\delta_c^j \delta_a^g - \delta_a^j \delta_c^g)\ \omega^{xc} \omega^{xg} e_{jma} \end{aligned}$$

$$= -\omega^{xc}\omega^{xg} I_{gd} e_{cmd} + \omega^{xc}\omega^{xc} I_{ad} e_{amd}$$

$$+ \omega^{xc}\omega^{xg} \frac{1}{2}(I_{11} + I_{22} + I_{33})\, e_{cmg} - \omega^{xc}\omega^{xc} \frac{1}{2}(I_{11} + I_{22} + I_{33}) e_{ama}$$

$$= -\omega^{xc}\omega^{xg} {}_{F}I_{gd} e_{cmd} \, . \qquad \text{(A18)}$$

Then, in conclusion,

$$T_F^{xm} = M_F r_F^{xd}(\dot{V}^{xj} + V^{xk}\omega^{xc} e_{jck}) e_{jmd}$$

$$+ {}_{F}I_{cm} \dot{\omega}^{xc} - \omega^{xc}\omega^{xg} {}_{F}I_{gd} e_{cmd} \qquad \text{(A19)}$$

which is obtained by substitution of (A15) and (A18) into (A11).

EXPANSION OF FORCE AND TORQUE EQUATIONS. Equation A8 is the force equation and equation A19 is the torque equation. The terms in these equations are defined as follows:

(a) M_F is the aircraft mass.

(b) V^{xi} are body axes velocities.

(c) ω^- is an angular velocity around the appropriate body axis.

(d) r_F^- is the displacement of the center of gravity from origin of the x-axes. Here $r_F^- = 0$.

(e) F_F^{xj} are body axes forces.

(f) T_F^{xm} are body axes torques.

Dropping the subscript F these equations are then written as:

$$F^{xj} = M(\dot{V}^{xj} + V^{xk}\omega^{xc} e_{jck})$$

$$T^{xm} = I_{cm}\dot{\omega}^{xc} - \omega^{xg} I_{gd} e_{cmd}\, \omega^{xc}$$

Expanding these equations there is:

$$F^{x1} = M(\dot{V}^{x1} + V^{x3}\omega^{x2} - V^{x2}\omega^{x3})$$

$$F^{x2} = M(\dot{V}^{x2} + V^{x1}\omega^{x3} - V^{x3}\omega^{x1})$$

$$F^{x3} = M(\dot{V}^{x3} + V^{x2}\omega^{x1} - V^{x1}\omega^{x2})$$

$$T^{x1} = I_{11}\dot{\omega}^{x1} + I_{13}\dot{\omega}^{x3} + \omega^{x2}\omega^{x1} I_{13} + \omega^{x2}\omega^{x3} I_{33} - \omega^{x3}\omega^{x2} I_{22}$$

$$T^{x2} = I_{22}\dot{\omega}^{x2} - \omega^{x1}\omega^{x1} I_{13} - \omega^{x1}\omega^{x3} I_{33} + \omega^{x3}\omega^{x1} I_{11} + \omega^{x3}\omega^{x3} I_{31}$$

$$T^{x3} = I_{33}\dot{\omega}^{x3} + I_{31}\dot{\omega}^{x1} - \omega^{x2}\omega^{x3} I_{31} - \omega^{x2}\omega^{x1} I_{11} + \omega^{x1}\omega^{x2} I_{22}$$

Since the aircraft is symmetric with the x1 - x3 plane only the I_{13} products of inertia exist. I_{13} carries a minus sign.

Finally, by using more conventional nomenclature these force and moment equations will be more readily recognized. F^{x1}, F^{x2}, F^{x3}, T^{x1}, T^{x2} and T^{x3} are the total external forces and torques, and as such denote aerodynamic effects (also gravity and thrust effects if applicable).

$F^{x1} = X_a$, total aerodynamic x-force.

$F^{x2} = Y_a$, total aerodynamic y-force.

$F^{x3} = Z_a$, total aerodynamic z-force.

$T^{x1} = L_a$, total aerodynamic rolling moment.

$T^{x2} = M_a$, total aerodynamic pitching moment.

$T^{x3} = N_a$, total aerodynamic turning moment.

$V^{x1} = U$, velocity along x-body axis.

$V^{x2} = V$, velocity along y-body axis.

$V^{x3} = W$, velocity along z-body axis.

$\omega^{x1} = p$, angular velocity around x-body axis.

$\omega^{x2} = q_1$, angular velocity around y-body axis.

$\omega^{x3} = r$, angular velocity around z-body axis.

$M = m$, the aircraft mass.

The equations for the basic aircraft then become the following expressions.

$$X_a = m(\dot{U} + Wq_1 - Vr)$$

$$Y_a = m(\dot{V} + Ur - Wp)$$

$$Z_a = m(\dot{W} + Vp - Uq_1)$$

$$L_a = I_{11}\dot{p} - I_{13}(\dot{r} + pq_1) + (I_{33} - I_{22})q_1 r$$

$$M_a = I_{22}\dot{q}_1 - I_{13}(r^2 - p^2) + (I_{11} - I_{33})pr$$

$$N_a = I_{33}\dot{r} - I_{13}(\dot{p} - q_1 r) + (I_{22} - I_{11})pq_1$$